MATH 126 PROBLEM SET 2: LINEAR ALGEBRA

This problem set is due Wednesday, October 4. All vector spaces are assumed to be finite dimensional.

1) If $v_1, \ldots, v_m$ are linearly independent elements of $\mathbb{F}_p^n$ show that there are $p^n - p^m$ elements of $w \in \mathbb{F}_p^n$ such that $v_1, \ldots, v_m, w$ are linearly independent. Deduce that $GL_n(\mathbb{F}_p)$ has $(p^n - 1)(p^n - p)\ldots(p^n - p^{n-1})$ elements.

2) If $\pi \in \mathrm{End}\,(V)$ and $\pi^2 = \pi$ show that $V = \ker \pi \oplus \mathrm{Im}\, \pi$.

3) If $V/\mathbb{C}$ is a vector space and $\alpha \in \mathrm{End}\,(V)$ satisfies $\alpha^m = 1$ for some positive integer m, show that α is diagonalisable.

4) Suppose that k is a field of characteristic $\neq 2$ and that V/k is a vector space with basis $\mathcal{B} = (v_1, v_2)$. Let $\mathcal{C}$ denote the basis $(v_1 \otimes v_1, v_1 \otimes v_2, v_2 \otimes v_1, v_2 \otimes v_2)$ of $V \otimes V$. Let $\mathcal{D}$ denote the basis $(2v_1 \otimes v_1, v_1 \otimes v_2 + v_2 \otimes v_1, 2v_2 \otimes v_2)$ of $S^2(V)$. Let $\mathcal{E}$ denote the basis $(v_1 \otimes v_2 - v_2 \otimes v_1)$ of $\wedge^2 V$.

a) If $[\alpha]_{\mathcal{B}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, find $[\alpha \otimes \alpha]_{\mathcal{C}}$, $[S^2(\alpha)]_{\mathcal{D}}$ and $[\wedge^2 \alpha]_{\mathcal{E}}$.

b) If $[\alpha]_{\mathcal{B}} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, find $[\alpha \otimes \alpha]_{\mathcal{C}}$ and $[S^2(\alpha)]_{\mathcal{D}}$.

c) If $[\alpha]_{\mathcal{B}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, find $[\wedge^2(\alpha)]_{\mathcal{E}}$.

5) Prove that there is a unique isomorphism $\alpha : V \otimes (W \oplus U) \xrightarrow{\sim} (V \otimes W) \oplus (V \otimes U)$ such that $\alpha(v \otimes (w, u)) = (v \otimes w, v \otimes u)$.

6) Prove that there is a unique isomorphism $\beta : V \otimes V^\vee \xrightarrow{\sim} \mathrm{End}\,(V)$ such that $\beta(v \otimes f)(w) = f(w)v$. Show also that there is a unique linear map $\gamma : V^\vee \otimes V \to k$ such that $\gamma(f \otimes v) = f(v)$. Moreover show that there is a unique linear map $\mu : \mathrm{End}\,(V) \otimes \mathrm{End}\,(V) \to \mathrm{End}\,(V)$ such that $\mu(f \otimes g) = fg$. Finally show that the composite

$$\mathrm{End}\,(V) \otimes \mathrm{End}\,(V) \xrightarrow{\beta^{-1} \otimes \beta^{-1}} V \otimes V^\vee \otimes V \otimes V^\vee \xrightarrow{1 \otimes \gamma \otimes 1} V \otimes V^\vee \xrightarrow{\beta} \mathrm{End}\,(V)$$

coincides with μ.

7) If $f \in \mathrm{End}\,(V)$ calculate the image of f under the composite

$$\mathrm{End}\,(V) \xrightarrow{\beta^{-1}} V \otimes V^\vee \cong V^\vee \otimes V \xrightarrow{\gamma} k$$

in terms of the matrix of f with respect to some basis of V.

8) If $f \in \mathrm{End}\,(V)$ and $g \in \mathrm{End}\,(W)$ show that $\det(f \otimes 1_W) = \det(f)^{\dim W}$ and hence that $\det(f \otimes g) = \det(f)^{\dim W} \det(g)^{\dim V}$.