MATH 126 PROBLEM SET 7: OPTIONAL

This problem set is not for credit, but to provide those of you bored over Thanksgiving with something to do. All groups are assumed to be finite and all vector spaces are assumed to be finite dimensional over an algebraicly closed field of characteristic 0.

You will need to know a little about cyclotomic fields. Recall that if ζ is a primitive n^{th} root of unity then $\mathbb{Q}(\zeta)/\mathbb{Q}$ is a Galois extension with Galois group isomorphic to $(\mathbb{Z}/n\mathbb{Z})^\times$ where $a \in (\mathbb{Z}/n\mathbb{Z})^\times$ corresponds to the automorphism of $\mathbb{Q}(\zeta)$ which sends ζ to ζ^a. Moreover the trace of ζ is 0 if the square of some prime divides n and -1 to the number of prime factors of n otherwise.

This problem set concerns representations ρ of a finite group G such that for each $g \in G$, $\rho(g)$ has at most two distinct eigenvalues. This will clearly be the case if $\dim \rho \leq 2$. If it is true for ρ then it is also true for $\rho \oplus \rho$. Thus one way to make the question precise is: classify finte groups G which have an irreducible faithful representation ρ such that $\dim \rho > 2$ and for all $g \in G$, $\rho(g)$ has at most two distinct eigenvalues. One famous finite group theorist told me he thought there were no such groups G. [I needed this result for some work I was doing about 10 years ago (see Duke Math. Journal 63 (1991), pp281-332, in particular lemma 9), which is why I came across the problem. It is easy to relax the restriction that G be finite, but the case G finite turns out to be the hardest case.]

Thus suppose that G has a faithful irreducible representation ρ such that $\dim \rho > 2$ and for all $g \in G$, $\rho(g)$ has at most two distinct eigenvalues. Let $Z(G)$ denote the centre of G and $\widetilde{G}$ the quotient $G/Z(G)$. For $g \in G - Z(G)$ let $\zeta_1(g), \zeta_2(g)$ denote the eigenvalues of $\rho(g)$ and $a_1(g), a_2(g)$ their multiplicities. Also let $\zeta(g) = \zeta_1(g)/\zeta_2(g)$. These are not well defined. Show that $\zeta(g) + \zeta(g)^{-1}$, $\operatorname{tr} \zeta(g)$, $a_1(g)a_2(g)$ and $(a_1(g) - a_2(g))^2$ are well defined functions on $\widetilde{G} - \{1\}$. Extend them to all of $\widetilde{G}$ by setting them to 2, 1, 0 and $(\dim \rho)^2$ respectively at 1. Show also that the order of $\zeta(g)$ makes sense and equals the order of $g \in \widetilde{G}$.

Show that

$$\#G(1 - (\dim \rho)^2) = \sum_{g \in G} a_1(g)a_2(g)(\zeta(g) + \zeta(g)^{-1} - 2)$$

and deduce that

$$\#\widetilde{G}((\dim \rho)^2 - 1) = 2 \sum_{g \in \widetilde{G}} a_1(g)a_2(g)(1 - \operatorname{tr} \zeta(g)/o(g)),$$

where $o(g)$ denotes the order of $g \in \widetilde{G}$. Now show that

$$7\#\widetilde{G} \leq -8 \sum_{g \in \widetilde{G}} \operatorname{tr} \zeta(g)/o(g).$$

[Hint: Consider $\dim\rho = 3$ separately.] Also show that if $o(g) = 1$ then $\operatorname{tr}\zeta(g)/o(g) = 1$, if $o(g) = 2$ then $\operatorname{tr}\zeta(g)/o(g) = -1$ and if $o(g) > 2$ then $|\operatorname{tr}\zeta(g)/o(g)| < 1/2$. Deduce that at least $3(\#\widetilde{G}+4)/4$ elements of $\widetilde{G}$ have order exactly 2.

If $g_1, g_2 \in \widetilde{G}$ satisfy $g_1^2 = g_2^2 = (g_1g_2)^2$ show that g_1 and g_2 commute. Hence show that if $g \in \widetilde{G}$ has order 2 then $\#Z_G(g) \geq \#\widetilde{G}/2 + 5$. Deduce that $g \in Z(\widetilde{G})$. Also deduce that $\widetilde{G}$ is an abelian group and that every element $g \in G$ squares to the identity. Deduce that G has a unique Sylow 2-subgroup G_2 and that $G \cong G_1 \times G_2$, where G_1 is a cyclic group of odd order.

If $\chi \in \widehat{\widetilde{G}}$ show that

$$\sum_{g\in\widetilde{G}} \chi(g)(a_1(g) - a_2(g))^2$$

is $\#\widetilde{G}$ if $\rho\otimes\chi \cong \rho$ and is zero otherwise. Deduce that

$$(a_1(g) - a_2(g))^2 = \sum \chi(g)$$

where the sum is over $\chi \in \widehat{\widetilde{G}}$ with $\rho\otimes\chi \cong \chi$. Further deduce that either $a_1(g) = a_2(g)$ or that

$$(a_1(g) - a_2(g))^2 = \#\{\chi \in \widehat{\widetilde{G}} :\ \rho\otimes\chi \cong \rho\} = (\dim\rho)^2.$$

Next deduce that if $g \notin Z(G)$ that $\operatorname{tr}\rho(g) = 0$ and that $(\dim\rho)^2 = \#\widetilde{G}$, i.e. $\dim\rho = 2^d$ and $\#\widetilde{G} = 2^{2d}$ for some $d > 1$.

Now let's work in the other direction and construct some pairs (G, ρ) as above. More specifically let G be a group such that $Z(G)$ is cyclic and $\widetilde{G} = G/Z(G)$ is an abelian group in which every element squares to the identity. For instance let Q denote the group $\langle a, b :\ a^4 = b^4 = a^2b^2 = a^3bab = 1\rangle$. Show that Q^d/H, where H is the subgroup generated by all elements which are a^2 in two of the d factors and 1 elsewhere, is an example of such a group G.

For $g_1, g_2 \in \widetilde{G}$ set $[g_1, g_2] = g_1g_2g_1^{-1}g_2^{-1} \in Z(G)$. Show that $[g_1g_2, g_3] = [g_1, g_3][g_2, g_3]$, $[g_1, g_1] = 1$, $[g_1, g_2] = [g_2, g_1]^{-1}$ and $[g_1, g_2]^2 = 1$ so that $[\ ,\ ]$ is valued in a cyclic group of order 2, $\{\pm 1\} \subset Z(G)$. Deduce that $G/\{\pm 1\}$ is abelian. Also show that if $[g_1, g_2] = 1$ for all $g_2 \in \widetilde{G}$ then $g_1 = 1$. Deduce that $\#\widetilde{G} = 2^{2d}$ for some $d \in \mathbb{Z}_{>0}$.

Deduce also that there is an abelian subgroup $H < G$ with $Z(G) < H$ and $\#G/H = 2^d$. If ψ is an irreducible representation of $Z(G)$ we may extend it to a representation $\widetilde{\psi}$ of H. Set $\rho_\psi = \operatorname{Ind}_H^G\widetilde{\psi}$. If $\psi(-1) = -1$ compute the character of ρ_ψ and show that it is irreducible. Show that G has $\#G/2$ one dimensional representations and $\#Z(G)/2$ irreducible representations of dimension 2^d. If ψ is a faithful character of $Z(G)$ show that ρ_ψ is an irreducible faithful representation of G such that for all $g \in G$, $\rho_\psi(g)$ has at most two distinct eigenvalues.