SYLLABUS FOR MATH 128, LIE ALGEBRAS
HARVARD UNIVERSITY, FALL 2002

Lie algebras appear in mathematics in many ways.

- They represent the local structure of *Lie groups*, groups with a differentiable structure.
- They represent infinitesimal actions on vector spaces, actions satisfying rules like the Leibniz rule $d(fg) = f\,dg + df\,g$.
- They represent the non-commutativity of an associative algebra.

We will be studying Lie algebras from several points of view:

- algebraic;
- combinatorical; and
- geometric.

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Prerequisites

You should know basic group theory and linear algebra, on the level of Math 122. Basic topology is helpful for understanding the connection with Lie groups, and more advanced linear algebra (tensor products, symmetric products, etc.) is very useful, but neither is a prerequisite for the course.

Assignments

There will be weekly problem sets starting on the second week of classes. They are due Wednesdays at the beginning of class. Late problem sets may not be graded, at the CA's discretion. Collaboration is encouraged, but please credit your collaborators and write up your solutions by yourself and in your own words.

Each student will be asked to give a short (half-hour) in-class presentation. Further information and suggested topics will be distributed in the second week of classes. The presentations will be spread out throughout the semester.

Date: September 15, 2002.

Textbooks and references

There are two optional texts for the class:

> *Representation Theory: A First Course*, by William Fulton and Joe Harris.
>
> *Complex Semisimple Lie Algebras*, by Jean-Pierre Serre.

The level of the books is generally high for an undergraduate course; I will explain the material in more elementary terms, as appropriate for the audience. They are on reserve at Cabot.

In addition, some books that may be useful for reference are

> *Lie Groups and Lie Algebras*, by Nicolas Bourbaki, particularly chapters 4–6.
>
> *The Classical Groups, their Invariants and Representations*, by Hermann Weyl.

In addition, the students will jointly produce a reference set of notes. This will be organised in the first week of classes.

Topics covered

The following topics may vary, depending on student background and interest.

Introduction, examples, and motivation. A *Lie group* is a group with some differentiable structure; for example, invertible 2×2 matrices. *Lie algebras* answer the question: What does a Lie group look like locally?

- Introduction to Lie algebras: GL_2 and $\mathfrak{gl}_2$.
- The cross product and 3-dimensional rotations.
- Lie group homomorphisms and invariance of the Lie bracket. (Fulton and Harris, §8.1)
- The classical series. Other examples of Lie algebras.
- Lie algebra representations and the meaning of the Jacobi relation.

Lie algebras in general. Before we can classify Lie algebras, we need some general tools. How is multiplication in a Lie group related to its Lie algebra? What is the Lie algebra with the minimal set of relations, and how large is it?

- Formal group laws and the Baker-Campbell-Hausdorff formula.
- Universal enveloping algebra.
- Free Lie algebras.
- The Poincaré-Birkhoff-Witt theorem.

Reducible Lie algebras. What do maps from one Lie algebra to another look like? When can we break a Lie algebra down into simpler pieces?

- The upper and lower central series.
- Abelian, nilpotent, solvable, simple, and semi-simple Lie algebras.

- Quotients and extensions. The radical.
- Levi decomposition.

Simple Lie algebras. After reducing a Lie algebra as much as possible, we are left with an irreducible core, a *simple Lie algebra*. These turn out not to be so simple: there are 4 infinite families and 5 exceptional cases. Curiously, similar *Dynkin diagrams* occur in many other contexts, for example in the classification of 3-dimensional polyhedra.

- $\mathfrak{sl}_2$ and its representations. (Fulton and Harris §11)
- Cartan subalgebras. Root vectors.
- $\mathfrak{sl}_3$ and its representations. (Fulton and Harris §12)
- Dynkin diagrams.
- The classification of simple Lie algebras.

The classical Lie algebras. The *classical Lie algebras* are the infinite families of simple Lie algebras in the classification above (as opposed to the exceptional ones). In some aspects, the four families can be reduced to just two families; we introduce *super vector spaces* (which have a negative dimension!) to do this.

- Invariant theory.
- Super vector spaces and Lie algebras.
- The ortho-symplectic family of Lie algebras.

Bonus topics. Lie algebras are still an active field of research. We illustrate some recent work and conjectures related to Lie algebras.

- Deligne's conjecture: a family of exceptional Lie algebras?
- Vogel's conjecture: a 2-parameter family including all simple Lie algebras?
- Lie algebras in 3-manifold topology.

Grading

Grading will be based on:

- Weekly problem sets (25% of the grade)
- Class participation, including note-taking (5%)
- An in-class presentation (20%)
- Take-home midterm (20%)
- Take-home final (subject to University approval) (30%)