# Math 128

Problem set 2

Feb. 13, 2002, due Feb 26

**1.** Let $H$ be the group of all real three by three matrices of the form

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

Consider the algebrea of all operators on functions of one real variable. Let $\mathbf{h}$ denote three dimensional space of operators spanned by the identity operator, $\mathbf{1}$, the operator $\mathbf{x}$ consisting of multiplication by $x$, and the operator $d/dx$:

$$\begin{aligned} \mathbf{1}: &\quad f \mapsto f \\ \mathbf{x}: &\quad f(x) \mapsto xf(x) \\ \frac{d}{dx}: &\quad f \mapsto \frac{df}{dx}. \end{aligned}$$

Show that $\mathbf{h}$ is a Lie subalgebra of the algebra of operators on smooth functions defined on $\mathbf{R}$. (It is called the Heisenberg algebra.) Show that $\mathbf{h}$ is isomorphic to the Lie algebra of $H$.

**2.** Let $Q$ be the $(n+2) \times (n+2)$ symmetric matrix

$$Q := \begin{pmatrix} 0 & 0 & 1 \\ 0 & I & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

where $I$ is the $n-$dimensional identity matrix. Define $g = o(1, n+1)$ to consist of all real $(n+2) \times (n+2)$ matrices $M$ which satisfy

$$MQ + QM^\dagger = 0.$$

Show that $g$ is a Lie algebra and describe its elements. In particular, show that $g$ has the structure of a *graded* Lie algebra:

$$g = g_{-1} \oplus g_0 \oplus g_1, \quad g_i = \{0\}, \quad i \neq -1, 0, 1, \quad [g_i, g_j] \subset g_{i+j},$$

where the subspaces $g_{-1}, g_0, g_1$ can be described as

$$\begin{aligned} g_1 &= \mathbf{R}^n \\ g_0 &= o(n) \oplus \mathbf{R} \\ g_1 &= \mathbf{R}^n \end{aligned}$$

where $g_0$ is the direct sum of the orthogonal algera, $o(n)$, in $n-$ dimensions and the trivial one dimensional algebra $\mathbf{R}$. The bracket of an element of $o(n)$ with an element of $g_{-1} = \mathbf{R}^n$ is the usual action of multipication of a vector by an (antisymmetric) matrix. The bracket of $r \in \mathbf{R}$ is just scalar multiplication by $r$. How do these elements act on $g_1$ which is also identified with $\mathbf{R}^n$?

**3.** In $\mathbf{R}^n$ we can identify the vector

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \text{ with } u_1 \frac{\partial}{\partial x_1} + \cdots + u_n \frac{\partial}{\partial x_n}$$

considered as a "constant" vector field, thought of as an "infinitesimal translation". The antisymmetric matrix $A = (a_{ij})$ is identified with the "linear" vector field

$$\sum_{i,j} a_{ij} x_i \frac{\partial}{\partial x_j},$$

thought of as an "infinitesimal rotation". Show that the set of all vector fields of these two types forms a Lie algebra isomorphic to the Lie algebra of the group of Euclidean motions in $n$- dimensions. If we throw in the scalar multiples of the Euler vector field

$$E := x_1 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n}$$

we get the algebra of infinitesimal *affine* conformal transformations, isomorphic to the subalgebra $g_{-1} \oplus g_0$ of the algebra $g$ in problem **2**. To understand $g_1$, consider the "conformal inversion" map

$$C : \; x \mapsto \frac{x}{|x|^2}.$$

(Strictly speaking this map is not defined at $x = 0$, or rather sends the origin, $x = 0$ into the "point at infinity". Let us blissfully ignore all such questions of domain of definition in the following computation.) Notice that $C^2$ =id. For each constant vector $u$ consider the one parameter group of transformations:

$$x \mapsto C(x + tu)C^{-1}.$$

Compute the corresponding vector field. You will find that it is defined everywhere and is "quadratic" in the sense that the coefficients of all the $\partial/\partial x_i$, i.e. all the components of this vector field are homogeneous quadratic polynomials. There is one for each $u$, so an $n-$dimensional family. Show that they behave like $g_1$ in that they, together with the preceeding (constant and linear) vector fields, span a Lie algebra isomorphic to $g = o(1, n+1)$ of problem **2**.

Since $C$ is a conformal map (it preserves angles but not lengths) the vector fields we have constructed are infinitesimal conformal transformations. It is a theorem (due to Liouville) that if $n \geq 3$ these are all the infinitesimal conformal transformations.

The algebra $o(2)$ acts irreducibly on the *real* plane $\mathbf{R}^2$ since no line through the origin is fixed by any rotation through angle other than a multiple of $\pi$. But it does *not* act irreducibly on the *complex* plane $\mathbf{C}^2$ since in the complex plane there are null vectors:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = i \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

for example. So acting irreducibly over the complex numbers is more restrictive.

Remember that all non-degenerate bilinear forms on a vector space of dimension $n$ over the complex numbers are the same, up to isomorphism. So we can talk of the algebra $o(n)$ consising of all $n \times n$ matrices satisfying

$$(Ax, y) + (x, Ay) = 0$$

where $(\ ,\ )$ is a non-degenerate bilinear form (*not* a scalar product: $(x, y) = (y, x)$ with no complex conjugation). Given a pair of vectors $x$ and $y$ we can define the transformation $A_{x,y}$ by

$$A_{x,y}u = (y, u)x - (x, u)y$$

and check that

$$(A_{x,y}u, v) + (u, A_{x,y}v) = 0.$$

**4.** Show that $o(n)$ acts irreducibly on $\mathbf{C}^n$ when $n \geq 3$. (You might want to use the fact that if $(e, e) = 0$ there is an $x$ with $(e, x) = 0$ and $(x, x) \neq 0$ which is not true for $n = 2$.)