Math. 128 Problem set 6.

Graphology 2, Coxeter graphs.

March 11, 2004, due March **25**

The Weyl group of a semi-simple Lie algebra is a finite subgroup of the (real) orthogonal group $O(E)$ generated by reflections. We will call such a subgroup a **reflection group**. If the algebra is simple, then the action of the Weyl group on E is irreducible. So we can pose the following problem: Classify all (finite) reflection groups which act irreducibly.

Let us begin by classifying such groups when E is the two dimensional Euclidean plane. The most general reflection through a line through the origin is given by

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}.$$

Indeed the columns of this matrix have length one and are orthogonal, so this is an orthogonal matrix, and

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$

So our matrix is reflection about the line through the origin which makes an angle θ with the x-axis.

Now if our group G contains only a single reflection there is nothing more to say. (In fact this group does not act irreducibly since it preserves the line of reflection.) If our group contains more than one reflection, we may take one of these reflections to be about the x-axis, and we may choose the line of a second reflection to be one which makes smallest angle with the x-axis. These two reflections must then generate the group. We have

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$$

which is rotation through angle 2θ. Since the group is finite we must have $2\theta = 2\pi/m$ for some integer $m \geq 2$ and the corresponding group is the dihedral group of order $2m$, the group of symmetries of a regular polygon with m sides. So we get an infinite family of finite subgroups of $O(2)$ generated by reflections.

Which of these are Weyl groups? Well, to be a Weyl group, G must preserve a lattice (the lattice in the plane generated by the roots of the Lie algebra). If we chose a basis of the plane consisting of vectors generating this lattice, then every element of the group, when expressed in terms of this basis must have integer entries. In particular, its trace must be an integer (and the trace is independent of the basis used). Applied to the rotation above we see that

$$2\cos 2\theta \quad \text{is an integer.}$$

Now $2\cos 2\pi/5$ is not an integer. The only possible values for m are 3 ($\cos 2\pi/3 = -\frac{1}{2}$), 4($\cos 2\pi/4 = 0$) and 6 ($\cos 2\pi/6 = \frac{1}{2}$). For all larger values of m we have $\frac{1}{2} < \cos 2\pi/m < 1$. (This result about which rotations can preserve a lattice goes back to Kepler.)

The allowed values correspond to the Weyl groups of $A_2, B_2 = C_2$ and G_2 in our classification of the simple Lie algebras.

From this analysis you might think that the classification of the irreducible reflection groups is much more complicated than the classification of the Weyl groups. In fact this is not the case. The Weyl group of B_n is isomorphic to the Weyl group of C_n, so the Weyl groups (for $n \geq 3$) are those of A_n, B_n, D_n and the Weyl groups of the exceptional algebras F_4, E_6, E_7 and E_8. It turns out that the classification of irreducible reflection groups for $n \geq 3$ involves only two additional cases, H_3 and H_4 in the notation below, in three and four dimensions. The purpose of this exercise set is to walk you through this fact.

Contents

1 Roots.

Let W denote a finite reflection group acting on a Euclidean space V. If $0 \neq \alpha \in V$ we let s_α denote the reflection through the hyperplane H_α perpendicular to α so

$$s_\alpha \lambda = \lambda - 2\frac{(\lambda, \alpha)}{(\alpha, \alpha)}\alpha.$$

If $s_\alpha \in W$ and $w \in W$ then $s_{w\alpha} = ws_\alpha w^{-1} \in W$. Thus W permutes the lines L_α through the vectors α such that $s_\alpha \in W$. Only the lines are determined by W. But we shall select a pair of non-zero vectors in each line. So we axiomatize the situation by defining a **root system** to be a finite set Φ of non-zero vectors satisfying

R1. If $\alpha \in \Phi$ the $r\alpha \in \Phi$ if and only if $r = \pm 1$ and
R2. If $\alpha, \beta \in \Phi$ then $s_\alpha \beta \in \Phi$.

Given a root system Φ we let $W(\Phi)$ denote the group generated by the $s_\alpha, \alpha \in \Phi$. Let V_Φ denote the subspace of V spanned by the $\alpha \in \Phi$. Then

$$V_\phi^\perp = \bigcap_{\alpha \in \Phi} H_\alpha$$

consists of vectors which are fixed by all $w \in W(\Phi)$ while if $w\alpha = \alpha$ for all $\alpha \in \Phi$ then w acts as the identity on V_ϕ as well. So we have an inclusion of W into the group of all permutations of the finite set ϕ which shows that $W(\Phi)$ is finite.

Any finite reflection group can be realized as $W(\Phi)$ possibly with many different choices of Φ. We may modify the lengths of the vectors of Φ provided that all vectors on any given W orbit have the same length.

For the case of Weyl groups we had the additional condition

R3. $2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$,

which we are definitely *not* assuming in our present study.

We define the **rank** of a root system to be the dimension of the space V_Φ spanned by the roots. If $W(\Phi) = W(\Phi')$ then Φ and Φ' have the same rank (since the roots of one differ by a non-zero multiple from the roots of the other) so we can talk of the rank of a reflection group W.

A root system Φ is called **reducible** if we can decompose Φ into a disjoint union of two non-empty subsets $\Phi = \Phi_1 \cup \Phi_2$ such that every element of Φ_1 is orthogonal to every element of Φ_2. If this happens, then

$$V_\Phi = V_{\Phi_1} \oplus V_{\Phi_2}$$

is an orthogonal decomposition invariant under $W(\Phi)$ and Φ_1 and Φ_2 are root systems with $W(\Phi) = W(\Phi_1) \times W(\Phi_2)$. In other words the action of $W(\Phi)$ is reducible.

Conversely, suppose we had an orthogonal decomposition $V = V_1 \oplus V_2$ invariant under W and such that the restriction W_i of W to each of these subspaces is non-trivial so that $W = W_1 \times W_2$. Suppose that Φ is a root system for W, i.e that $W = W(\Phi)$. I claim that Φ is reducible. Indeed, set

$$\Phi_1 := \{\alpha \in \Phi | s_\alpha \in W_1\}, \quad \Phi_2 := \{\beta \in \Phi | s_\beta \in W_2\}.$$

The sets Φ_1 and Φ_2 satisfy our two axioms for a root system. If $\alpha \in \Phi_1$ and $\beta \in \Phi_2$ then s_α and s_β commute which implies that $s_\alpha \beta = \pm\beta$. From the formula for s_α we see that this forces $(\alpha, \beta) = 0$.

Now suppose that $\gamma \in \Phi$ so that $s_\gamma = (s_1, s_2) \in W_1 \times W_2 = W$. Let H_γ be the hyperplane orthogonal to γ. This consists of all vectors fixed by s_γ implying that either s_1 or s_2 is the identity. So $\Phi = \Phi_1 \cup \Phi_2$.

In short, $W(\Phi)$ is reducible if and only if Φ is reducible. We are interested in classifying the irreducible W. So we will have done this if we can classify all the irreducible Φ where we will choose Φ so that all the vectors in Φ have length one.

Before getting to this go back and read sections **5.6-5.8** of the notes and convince yourself that all the results of those sections hold for our more general definition of a root system. In particular we know that we can choose a base Δ of Φ and hence the corresponding system of **simple reflections** which generate W (see the end of section **5.7**).

2 Generators, relations and Coxeter graphs.

For any $\alpha, \beta \in \Phi$ let $m(\alpha, \beta)$ denote the order of $s_\alpha s_\beta$. This is the smallest integer m such that $(s_\alpha s_\beta)^m = 1$(the identity element of $W(\Phi)$). For example $m(\alpha, \alpha) = 1$. If W is the dihedral group of order $2m$ in the plane, and α and β are such that the lines H_α and H_β make the smallest non-zero angle, then we know that $s_\alpha s_\beta$ has order m. Furthermore we know in this case that the dihedral group is generated by s_α and s_β subject only to the condition that $(s_\alpha s_\beta)^m = 1$. The following theorem is a generalization of this example:

Theorem 1 *Fix a base Δ of Φ and hence a set of simple reflections $\{s_\alpha, \alpha \in \Delta\}$. Then $W = W(\Phi)$ is generated by the $s_\alpha,\ \alpha \in \Delta$ subject only to the relations*

$$(s_\alpha s_\beta)^{m(\alpha,\beta)} = 1.$$

Proof. We already know that the $s_\alpha,\ \alpha \in \Delta$ generate W. We must show that any relation of the form

$$s_1 s_2 \dots s_r = 1 \tag{1}$$

among these generators is a consequence of the relations listed in the theorem. In an equation such as (1) the s_i stand for $s_{\alpha_i},\ \alpha_i \in \Delta$ (repetitions allowed).

Notice that in (1) r must be even since $\det s_\alpha = -1$. If $r = 2$ then the equation $s_1 s_2 = 1$ implies that $s_1 = s_2$ since $s_1^2 = 1$. But the equation $s_1^2 = 1$ is one of the relations stated in the theorem. The strategy is to proceed by induction on $r = 2q$ to show that (1) can be rewritten in terms of the relations of the theorem. For this we will make repeated use of the equation

$$\ell(w) = n(w)$$

of section **5.8** where $\ell(w)$ is the length of any shortest expression of w as a product of the $s_\alpha,\ \alpha \in \Delta$ and $n(w)$ is the number of positive roots made negative by w. In particular, if $\alpha \in \Delta$ and $w\alpha$ is positive then ws_α takes all the positive roots that w makes negative into negative roots, and in addition makes α negative. We write this as

$$w\alpha > 0,\ \alpha \in \Delta \Rightarrow n(ws_\alpha) = n(w) + 1. \tag{2}$$

We now state a crucial proposition which was implicit in the Lie algebra notes but I want to make explicit here in order to prove the preceding theorem.

2.1 Deletion and exchange.

Proposition 1 *Let $w = s_1 s_2 \cdots s_k$ where $s_i = s_{\alpha_i}$, $\alpha_i \in \Delta$ and suppose that $n(w) < k$. Then there exist indices $1 \leq i < j \leq k$ such that*

1. $\alpha_i = (s_{i+1} s_{i+2} \cdots s_j)\alpha_j$,
2. $s_{i+1} s_{i+2} \cdots s_j = s_i s_{i+1} \cdots s_{j-1}$,
3. $w = s_1 \cdots s_{i-1} \hat{s}_i s_{i+1} \cdots \hat{s_j} \cdots s_k$

where the hat denotes omission.

Proof of the Proposition. 1. Starting with s_1 we must have some j such that $s_1 \cdots s_{j-1}\alpha_j$ is negative, for otherwise repeated application of (2) will imply that $n(w) = k$. Since α_j is positive, we must have some $1 \leq i < j$ such that $s_i(s_{i+1} \cdots s_{j-1})\alpha_j < 0$ while $(s_{i+1} \cdots s_{j-1})\alpha_j > 0$. We know that s_i changes only one positive root into a negative root and that root is α_i. This proves item 1.

2. From part 1) it follows that

$$(s_{i+1} s_{i+2} \cdots s_{j-1}) s_j (s_{i+1} s_{i+2} \cdots s_{j-1})^{-1} = s_i$$

which after multiplying both sides by $(s_{i+1} s_{i+2} \cdots s_j)$ gives item 2.

3. Multiply both sides of the equation in item 2 on the right by s_j to obtain

$$s_{i+1} s_{i+2} \cdots s_{j-1} = s_i s_{i+1} \cdots s_{j-1} s_j$$

and substitute this into the original expression for w. QED

Notice that as a corollary we deduce that $\ell(w) = n(w)$. Indeed if $n(w)$ is less than the length of any expression for w we can delete two entries in the expression for w.

2.2 Proof of the theorem.

To illustrate the induction idea suppose that $q = 2$ so that we have an equation

$$s_1 s_2 s_3 s_4 = 1.$$

We can write this as

$$s_1 s_2 s_3 = s_4$$

so the left hand side is not a reduced expression. Hence we can find $1 \leq i < j \leq 3$ which we may delete from the left hand side. In other words, from item 3) of the proposition we have either

$$s_4 = s_1, \quad s_4 = s_3, \quad \text{or } s_4 = s_2.$$

In the first case we can multiply on the left by $s_1 = s_4$ so $s_2 s_3 = 1$ and so the equation is the consequence of the relations $s_1^2 = 1$ and $s_2^2 = 1$. Similarly in the second case. In the third case our equation becomes

$$s_1 s_2 s_3 s_2 = 1.$$

Multiply on the right and left by s_2 to obtain

$$s_2 s_1 s_2 s_3 = 1.$$

Repeat the same procedure, i.e. multiply on the right by s_3 to obtain

$$s_2 s_1 s_2 = s_3.$$

So either $s_3 = s_2$ and hence $s_1 = s_2$ so the equation is $s_1^4 = 1$ or $s_3 = s_1 \neq s_2$ in which case our original equation is $(s_1 s_2)^2 = 1$ implying that $m(\alpha_1, \alpha_2) = 2$ and the equation is one of the relations of the theorem.

We now turn to the general induction argument. Write equation (1) as

$$s_1 \cdots s_{q+1} = s_r \cdots s_{q+2}.$$

The length of the element on the right is at most $q - 1$. By part 2) of the Proposition, we can find indices $1 \leq i < j \leq q + 1$ such that

$$s_{i+1} \cdots s_j = s_i \cdots s_{j-1}$$

which implies that

$$s_i \cdots s_{j_1} s_j \cdots s_{i+1} = 1.$$

If this involves fewer that r simple reflections, our induction hypothesis says that it can be derived from the relations of the theorem. Then we can replace $s_{i+1} \cdots s_j$ by $s_i \cdots s_{j-1}$ in (1) and write it as

$$s_1 \cdots s_i (s_i \cdots s_{j_1}) s_{j+1} \cdots s_r = s_1 \cdots \hat{s_i} \cdots \hat{s_j} \cdots s_r = 1$$

which involves two fewer reflections and hence is a consequence of the relation of the theorem by induction. This procedure will work unless $i = 1$ and $j = q + 1$ in which case the relation becomes

$$s_2 \cdots s_{q+1} = s_1 \cdots s_q$$

which involves r reflections so induction will not help. So let's rewrite (1) as

$$s_2 \cdots s_r s_1 = 1.$$

We can apply the same procedure as above, moving the last $q-1$ factors over to the right and applying part 2) of the Proposition. This will be successful unless the result of part 2) is

$$s_3 \cdots s_{q+1} = s_2 \cdots s_{q+1}$$

which we can rewrite as

$$s_3(s_2 s_3 \cdots s_{q+1}) s_{q+2} s_{q+1} \cdots s_4 = 1.$$

Again we could try part 2) of the Proposition to conclude the theorem, and this will work unless part 2) of the proposition yields

$$s_2 \cdots s_{q+1} = s_3 s_2 s_3 \cdots s_q.$$

But we also know that

$$s_2 \cdots s_{q+1} = s_1 \cdots s_q.$$

This means that we must have $s_1 = s_3$. Now again cyclically permute the left hand side of (1) and apply the above procedure which will work unless $s_2 = s_4$. Keep going. We will have reduced (1) to being a consequence of an equation with fewer elements (and hence by induction a consequence of the relations in the theorem) unless

$$s_1 = s_3 = \cdots = s_{r-1} \quad \text{and} \quad s_2 = s_4 = \cdots = s_r.$$

So (1) is of the form $(s_\alpha s_\beta)^q = 1$ which is a consequence of the relations of the theorem. QED

2.3 Coxeter graphs.

If $m(\alpha,\beta) = 2$ then $\alpha \perp \beta$ and conversely. We know that the group W is completely determined by the integers $m(\alpha,\beta)$ and we are interested in the irreducible case. So let us construct a graph whose vertices are in one to one correspondence with elements of Δ and where two vertices are connected by an edge if and only if $m(\alpha,\beta) \geq 3$, and label the edge by $m(\alpha,\beta)$. Actually, if $m(\alpha,\beta) = 3$ we simply omit the label 3.

From the theorem we know that the Coxeter graph determines a presentation of W by generators and relations. But we can be more precise. Let $W_i,\ i = 1,2$ be a finite reflection group acting on $V_i,\ i = 1,2$ and suppose that $V_{\Phi_i} = V_i$ for $i = 1,2$. So if we have chosen a base for each root system then the elements of each base form a basis of V_i. Also, let us choose our root systems so that all root vectors have unit length.

Proposition 2 *If W_1 and W_2 have the same Coxeter graph then there the map of V_1 into V_2 sending the root of Δ_1 into the root of Δ_2 corresponding to the same vertex in the Coxeter graph extends to an isometry of V_1 onto V_2 which induces an isomorphism of W_1 onto W_2.*

Proof. We know that the elements of Δ_1 form a basis of V_1 and the elements of Δ_2 form a basis of V_2. So the map sending each element of Δ_1 into the corresponding element of V_2 ext ends uniquely to a linear map of V_1 onto V_2. If $\alpha \neq \beta \in \Delta_1$ the angle θ between them is $\pi - \pi/m(\alpha, \beta)$ and since α and β are unit vectors

$$(\alpha, \beta) = \cos\theta = -\cos\frac{\pi}{m(\alpha, \beta)}.$$

The same calculation applies to the images of these under our map which proves that the map is an isometry and hence induces an isomorphism of W_1 with W_2.

So we define a **Coxeter graph** to be a finite undirected graph whose edges are labeled by integers ≥ 4, an edge without a label is understood to have the label 3. To each pair of vertices α and β we associate an integer $m(\alpha, \beta)$ where $m(\alpha, \alpha) = 1$, where $m(\alpha, \beta) = 2$ if α and β are not the vertices of an edge, and where, if α and β *are* joined by an edge then $m(\alpha, \beta)$ is the integer associated to that edge.

2.4 The matrix of a Coxeter graph.

To each Coxeter graph with n vertices we associate a symmetric $n \times n$ matrix whose rows and columns are labeled by the vertices of the graph and whose entry at the α, β position is

$$A(\alpha, \beta) = -\cos\frac{\pi}{m(\alpha, \beta)}.$$

These matrices are not the same as the matrices we studied in the last problem set.

Recall that any symmetric $n \times n$ matrix A defines a bilinear form on $\mathbb{R}^n$ sending a pair of vectors x, y into $x^t A y$ (where x^t is the row vector which is the transpose of x). The eigenvalues of a symmetric matrix are real, and the matrix is **positive definite**, i.e. satisfies $x^t A x > 0$ for all $x \neq 0$ if all its eigenvalues are positive. A convenient way of checking whether a symmetric matrix is positive definite is as follows: The **principal minors** of A are the determinants of the submatrices obtained by deleting the last k rows and columns of A. Then a symmetric matrix is positive definite if and only if all its principal minors are positive.

For example, the two by two matrix

$$\begin{pmatrix} 1 & -\cos\theta \\ -\cos\theta & 1 \end{pmatrix}$$

is positive definite if $\theta \neq 0$ or π.

If our Coxeter graph is the Coxeter graph of a finite reflection group then its associated matrix is positive definite since it represents the standard Euclidean metric in terms of the basis Δ whose elements are of unit length but not necessarily orthogonal. So our classification problem for finite reflection groups becomes: Find all connected Coxeter graphs whose associated matrix is positive definite. It will turn out that each of these is in fact the Coxeter graph of a finite reflection group. We will say the a Coxeter graph is positive definite if its associated matrix is positive definite.

3 A list of positive definite Coxeter graphs.

To prove that the graphs in Figure 1 are positive definite we should compute all the principal minors of each. But if we remove an appropriate vertex from any graph on the list we get another graph on the list. So it is enough to show that the determinants of the matrices associated to each of these graphs is positive.

It is easier to compute $\det 2A$ to get rid of the 2's that occur in the denominator of the cosines. For $n \geq 3$ notice that it is possible to label the vertices so that the vertex labeled n is connected to exactly one other vertex labeled $n-1$ and the edge joining them is labeled 3 (i.e. no label).

So if we let d_{n-1} denote the principal $(n-1)$st minor of $2A$ and d_{n-2} the principal $(n-2)$nd principal minor then the bottom row and the right hand column of $2A$ has a

$$-2\cos\frac{\pi}{3} = -1$$

in the $(n-1)$st position if the label of the last edge is 3, and a 2 in the n-th position and all zeros 0's elsewhere. So expanding $\det 2A$ along the bottom row gives

$$d_n = \det 2A = 2d_{n-1} - d_{n-2}.$$

So, for example, if we look at graphs of type A_n we see that $d_1 = 2, d_2 = 3$ and then inductively that $d_n = n+1$, proving that the graphs of type A_n are positive definite.

For graphs of type B_2 the determinant of $2A$ when $n = 2$ is 2, and then for $n = 3$ (taking the leftmost vertex to have label 3) we have

$$2A = \begin{pmatrix} 2 & -\sqrt{2} & 0 \\ -\sqrt{2} & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

whose determinant is 2 and then inductively (taking the leftmost vertex to have label n) that

$$\det 2A = 2.$$

1. Evaluate $\det 2A$ for each of the cases $D_n,\ E_6,\ E_7, E_8,\ F_4,\ H_3$, and H_4. For the last two you will want to make use of the fact that

$$\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4}.$$

The rest of this problem set is devoted to proving that these are the only positive definite connected Coxeter graphs. The strategy will be similar to

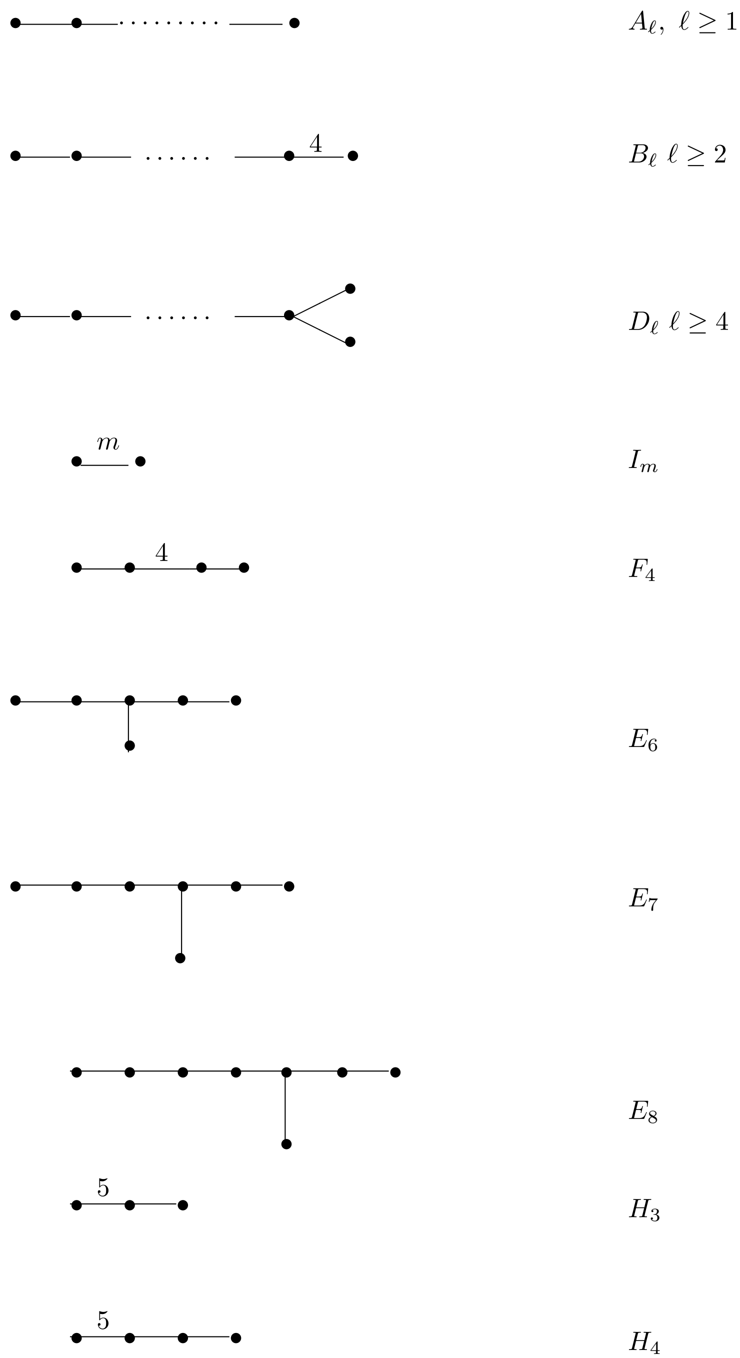
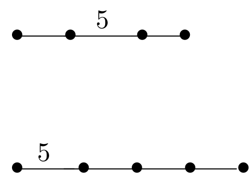


Figure 1: Positive definite Coxeter graphs.



Z_4

Z_5

Figure 2: Z_4 and Z_5 have negative determinant.

that of the last problem set. We will need a different version of the Perron-Frobenius theorem - this time for positive semi-definite matrices. Remember that the matrices we are currently considering have some negative entries and are symmetric. The matrices in the last problem set had non negative entries and did not have to be symmetric. Before stating the version of Perron-Frobenius that we need, we (you will be asked to) do a computation that we will need. Consider the two graphs Z_4 and Z_5 in Figure 2:

2. Compute $\det 2A$ for each of the graphs Z_4 and Z_5. In particular, verify that these determinants are negative.

4 A version of Perron-Frobenius for positive semi-definite matrices.

A symmetric matrix A is called **positive semi-definite** if all its eigenvalues are non-negative. What is the same, this says that $x^tAx \geq 0$ for all x. A convenient way of verifying this condition is to check that all its principal minors are non-negative. A matrix is called **indecomposable** if there is no way of breaking up the index set into two non-empty subsets I and J such that $a_{ij} = 0$ when $i \in I$ and $j \in J$. Here is the version of Perron-Frobenius that we will use.

Theorem 2 *Let A be a real symmetric matrix which is positive semi-definite, indecomposable, and such that $a_{ij} \leq 0$ when $i \neq j$. Then*

- *The set $N : \{x | x^tAx = 0\}$ coincides with the null space of A, i.e. the subspace consisting of all x such that $Ax = 0$,*
- *The dimension of N is at most one, and*
- *The smallest eigenvalue of A has multiplicity one and has an eigenvector all of whose coordinates are positive.*

Proof. Clearly the null space of A is contained in N. We need to show the reverse inclusion. Since A is symmetric, we can find an orthogonal matrix O such that $O^tAO =: D$ is diagonal with non-negative entries $d_1, \ldots d_n$ along the

diagonal. If $x^t A x = 0$, then setting $y = O^{-1}x = O^t x$ we get $y^t D y = 0$ which says that
$$\sum d_i y_i^2 = 0.$$
Since all the d_i are non-negative, this can hold only if $y_i = 0$ whenever $d_i > 0$ which implies that $Dy = 0$ which is equivalent to $Ax = 0$. This proves the first item.

Suppose that there is a non-zero vector $x \in N$. Let $z = |x|$. Recall that this means that the coordinates of z are $|x|_i$. Since $a_{ij} \leq 0$ for $i \neq j$,
$$0 \leq z^t A z \leq x^t A x = 0$$
so $z = |x| \in N$. We claim that all the coordinates of z are strictly positive. Indeed, let J be the set of indices for which $z_j > 0$ and let I be its complement. We know that $J \neq \emptyset$. We know that $Az = 0$ from the first item, which says that
$$\sum_j a_{ij} z_j = 0$$
for all i. In this sum, we need only sum over $j \in J$ since the remaining coordinates of z are 0. But since $z_j > 0$ for $j \in J$ and all the $a_{ij} \leq 0$ for $i \neq j$, if there were some $i \notin J$ the only way for the above sum to vanish would be for all the $a_{ij} = 0$. Since A is indecomposable, this is precluded. So if $N \neq \{0\}$ then N contains a vector all of whose entries are positive. The argument also shows that for any $0 \neq x \in N$, all the entries of x are non-zero. But this implies that if we had two elements $x \neq 0$ and x' in N, and if we multiplied x by a factor r so that its first coordinate coincided with the first factor of x', then we must have $x' = rx$. This proves the second item of the theorem.

Now let d be the smallest eigenvalue of A (possibly 0). Then $A - dI$ is still symmetric, positive semi-definite and irreducible. The argument proving the second item applies to $A - dI$ proving the third item. QED

Here is how we are going to use the theorem: If Γ is a connected Coxeter graph, we say that Γ' is a **subgraph** if Γ' is obtained from Γ by omitting some vertices and adjacent edges, or by decreasing some labels or by both. We claim that the following is a corollary to the theorem:

Proposition 3 *Let Γ be a connected positive semi-definite Coxeter graph. Then every proper subgraph of G is positive definite.*

Proof. Let Γ' be a proper subgraph of Γ and let A' and A be the associated matrices. (Here the size of A' may be smaller than the size of A.) We have $m'_{ij} \leq m_{ij}$ so
$$a'_{ij} = -\cos\frac{\pi}{m'_{ij}} \geq -\cos\frac{\pi}{m_{ij}} = a_{ij}.$$
Suppose that A' is of size $k \times k$ and fails to be positive definite. Then there is some non-zero $x \in \mathbb{R}^k$ with $x^t A x \leq 0$. We may consider x as a vector in $\mathbb{R}^n$ by inserting 0's in the missing positions and also inserting 0's as rows an columns

in these positions to think of A' as an $n \times n$ matrix. Then since A is positive semi-definite we have

$$0 \leq |x|^t A|x| = \sum_{i,j} a_{ij}|x|_i|x|_j \leq \sum_{i,j} a'_{ij}|x|_i|x|_j.$$

But since $a'_{ij} \leq 0$ for $i \neq j$ we have

$$\sum_{i,j} a'_{ij}|x|_i|x|_j \leq \sum_{i,j} a'_{ij}x_ix_j \leq 0.$$

We get

$$0 \leq |x|^t A|x| \leq 0$$

so $A|x| = 0$ and we know from the theorem that all the coordinates of our enlarged x are non-zero so $k = n$. Also, since all the coordinates of $|x|$ are strictly positive, the only way that

$$\sum_{i,j} a_{ij}|x|_i|x|_j \leq \sum_{i,j} a'_{ij}|x|_i|x|_j$$

can fail to be a strict inequality is for $a_{ij} = a'_{ij}$ contradicting the hypothesis that Γ' is a proper subgraph of Γ. This proves the proposition.

3. Prove that H_3 and H_4 are the only positive definite connected Coxeter graphs with three or more vertices and with an edge label ≥ 5.

5 Extended Dynkin diagrams as positive semi-definite Coxeter graphs.

Consider the Coxeter graphs in Figure 3. We claim that each of these graphs is positive semi-definite but not positive definite. Since we get a graph in figure 1 if we remove a suitable vertex from each of these graphs, all we have to check is that each of them has determinant 0. For type $\tilde{A}_\ell$, $\ell \geq 2$ we really know this already. Indeed, the matrix of such a Coxeter graph is exactly $A = I - \frac{1}{2}B$ where B is the adjacency matrix of the graph, and we know that B has eigenvalue 2. for $\tilde{A}_2$ the matrix is

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

which has determinant 0.

Proposition 3 then implies that no connected positive semi-definite connected Coxeter graph other that $\tilde{A}_\ell$ can contain a circuit.

4. show that all the remain graphs in figure 3 also have determinant zero. [Hint: Use the inductive formula we used above to compute $\det 2A$.]

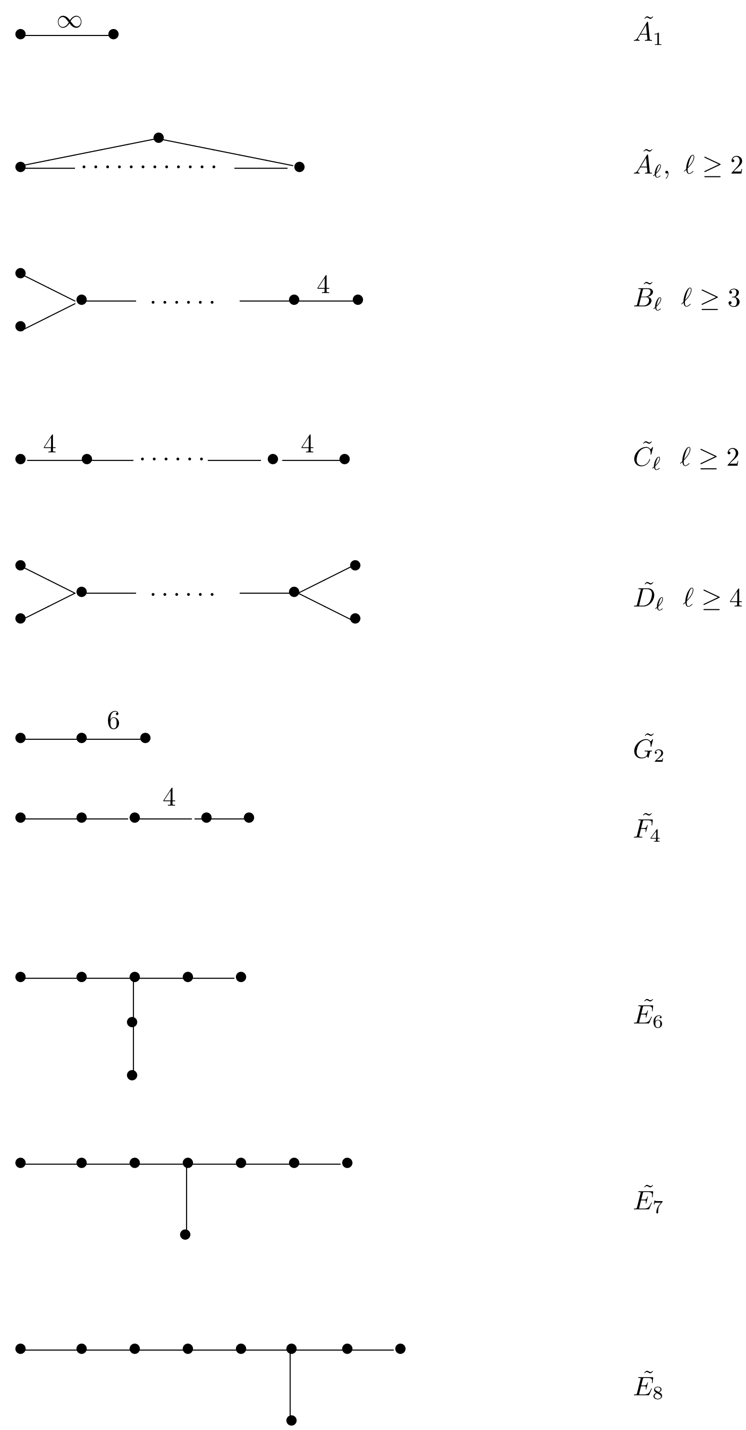


Figure 3: Some positive semi-definite Coxeter graphs.

The next problems will show that the graphs in Figure 1 and Figure 3 are all the positive semi-definite connected Coxeter graphs, and hence that the graphs in Figure 1 are all the positive definite connected Coxeter graphs. So suppose that Γ is a connected positive semi-definite Coxeter graph. Let n denote the number of vertices and let m denote the maximal label. We wish to show that no Γ which does not appear in Figure 1 or Figure 3 exists.

5. Suppose $m = 3$. Since A_n *is* on our list, Γ must contain a branch point. Show that Γ can not contain two branch points and that at most three edges meet at a branch point. [Hint: Use $\tilde{D}_\ell$ and $\tilde{D}_4$.] Conclude that no such Γ exists.[Hint: Use $\tilde{E}_6, \tilde{E}_7$ and $\tilde{E}_8$.]

We now must consider the case where $m \geq 4$.

6. Show that if ≥ 4 then at most one edge has a label > 3 and that there is no branch point.

Suppose that $m = 4$. Since $\Gamma \neq B_n$ both end edges are labeled 3.

7. Show that this case can not exist. [Hint: Use F_4 and $\tilde{F}_4$.]
So $m \geq 5$.

8. Complete the proof that no such Γ exists.