Lie Algebras

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Chapter 9

Clifford algebras and spin representations.

9.1 Definition and basic properties

9.1.1 Definition.

Let $\mathbf{p}$ be a vector space with a symmetric bilinear form $(\ ,\ )$. The **Clifford algebra** associated to this data is the algebra

$$C(\mathbf{p}) := T(\mathbf{p})/I$$

where $T(\mathbf{p})$ denotes the tensor algebra

$$T(\mathbf{p}) = k \oplus \mathbf{p} \oplus (\mathbf{p} \otimes \mathbf{p}) \oplus \cdots$$

and where I denotes the ideal in $T(\mathbf{p})$ generated by all elements of the form

$$y_1 y_2 + y_2 y_1 - 2(y_1, y_2)\mathbf{1}, \quad y_1,\ y_2\ \in \mathbf{p}$$

and $\mathbf{1}$ is the unit element of the tensor algebra. The space $\mathbf{p}$ injects as a subspace of $C(\mathbf{p})$ and generates $C(\mathbf{p})$ as an algebra.

A linear map f of $\mathbf{p}$ to an associative algebra A with unit 1_A is called a **Clifford map** if

$$f(y_1)f(y_2) + f(y_2)f(y_1) = 2(y_1, y_2)1_A, \quad \forall y_1,\ y_2\ \in \mathbf{p}$$

or what amounts to the same thing (by polarization since we are not over a field of characteristic 2) if

$$f(y)^2 = (y, y)1_A \ \ \forall\ y \in \mathbf{p}.$$

Any Clifford map gives rise to a unique algebra homomorphism of $C(\mathbf{p})$ to A whose restriction to $\mathbf{p}$ is f. The Clifford algebra is "universal" with respect to this property.

If the bilinear form is identically zero, then $C(\mathbf{p}) = \wedge\mathbf{p}$, the exterior algebra. But we will be interested in the opposite extreme, the case where the bilinear form is non-degenerate.

9.1.2 Gradation.

The ideal I defining the Clifford algebra is not $\mathbf{Z}$ homogeneous (unless the bilinear form is identically zero) since its generators $y_1y_2 + y_2y_1 - 2(y_1, y_2)\mathbf{1}$ are "mixed", being a sum of terms of degree two and degree zero in $T(\mathbf{p})$. But these terms are both even. So the $\mathbf{Z}/2\mathbf{Z}$ gradation *is* preserved upon passing to the quotient. In other words, $C(\mathbf{p})$ is a $\mathbf{Z}/2\mathbf{Z}$ graded algebra:

$$C(\mathbf{p}) = C_{\mathbf{0}}(\mathbf{p}) \oplus C_{\mathbf{1}}(\mathbf{p})$$

where the elements of $C_{\mathbf{0}}(\mathbf{p})$ consist of sums of products of elements of $\mathbf{p}$ with an even number of factors and $C_{\mathbf{1}}(\mathbf{p})$ consist of sums of terms each a product of elements of $\mathbf{p}$ with an odd number of factors. The usual rules for multiplication of a graded algebra obtain:

$$C_{\mathbf{0}}(\mathbf{p}) \cdot C_{\mathbf{0}}(\mathbf{p}) \subset C_{\mathbf{0}}(\mathbf{p}),\ C_{\mathbf{0}}(\mathbf{p}) \cdot C_{\mathbf{1}}(\mathbf{p}) \subset C_{\mathbf{1}}(\mathbf{p}),\ C_{\mathbf{1}}(\mathbf{p}) \cdot C_{\mathbf{1}}(\mathbf{p}) \subset C_{\mathbf{0}}(\mathbf{p}).$$

9.1.3 $\wedge\mathbf{p}$ as a $C(\mathbf{p})$ module.

Let $\mathbf{p}$ be a vector space with a non-degenerate symmetric bilinear form. The exterior algebra, $\wedge\mathbf{p}$ inherits a bilinear form which we continue to denote by $(\ ,\ )$. Here the spaces $\wedge^k(\mathbf{p})$ and $\wedge^\ell(\mathbf{p})$ are orthogonal if $k \neq \ell$ while

$$(x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_k) = \det\left((x_i, y_j)\right).$$

For $v \in \mathbf{p}$ let $\epsilon(v) \in \mathrm{End}(\wedge\mathbf{p})$ denote exterior multiplication by v and $\iota(v)$ be the transpose of $\epsilon(v)$ relative to this biinear form on $\wedge\mathbf{p}$.

So $\iota(v)$ is interior multiplication by the element of $\mathbf{p}^*$ corresponding to v under the map $\mathbf{p} \to \mathbf{p}^*$ induced by $(\ ,\ )_{\mathbf{p}}$. The map

$$\mathbf{p} \to\ \mathrm{End}(\wedge\mathbf{p}), \quad v \mapsto \epsilon(v) + \iota(v)$$

is a Clifford map, i.e. satisfies

$$(\epsilon(v) + \iota(v))^2 = (v, v)_{\mathbf{p}}\mathrm{id}$$

and so extends to a homomorphism of

$$C(\mathbf{p}) \to \mathrm{End}\wedge\mathbf{p}$$

making $\wedge\mathbf{p}$ into a $C(\mathbf{p})$ module. We let xy denote the product of x and y in $C(\mathbf{p})$.

9.1.4 Chevalley's linear identification of $C(\mathbf{p})$ with $\wedge\mathbf{p}$.

Consider the linear map

$$C(\mathbf{p}) \to \wedge\mathbf{p}, \qquad x \mapsto x1$$

where $1 \in \wedge^0\mathbf{p}$ under the identification of $\wedge^0\mathbf{p}$ with the ground field. The element $x1$ on the extreme right means the image of 1 under the action of $x \in C(\mathbf{p})$.

For elements $v_1, \ldots, v_k \in \mathbf{p}$ this map sends

$$\begin{array}{rcl} v_1 & \mapsto & v_1 \\ v_1v_2 & \mapsto & v_1 \wedge v_2 + (v_1, v_2)1 \\ v_1v_2v_3 & \mapsto & v_1 \wedge v_2 \wedge v_3 + (v_1, v_2)v_3 - (v_1, v_3)v_2 + (v_2, v_3)v_1 \\ v_1v_2v_3v_4 & \mapsto & v_1 \wedge v_2 \wedge v_3 \wedge v_4 + (v_2, v_3)v_1 \wedge v_4 - (v_2, v_4)v_1 \wedge v_3 \\ & & +(v_3, v_4)v_1 \wedge v_2 + (v_1, v_2)v_3 \wedge v_4 - (v_1, v_3)v_1 \wedge v_4 \\ & & +(v_1, v_4)v_2 \wedge v_3 + (v_1, v_4)(v_2, v_3) - (v_1, v_3)(v_2, v_4) + (v_1, v_2)(v_3, v_4) \\ \vdots & & \vdots \end{array}$$

If the v's form an "orthonormal" basis of $\mathbf{p}$ then the products

$$v_{i_1} \cdots v_{i_k}, \quad i_1 < i_2 \cdots < i_k, \ k = 0, 1, \ldots, n$$

form a basis of $C(\mathbf{p})$ while the

$$v_{i_1} \wedge \cdots \wedge v_{i_k}, \quad i_1 < i_2 \cdots < i_k, \ k = 0, 1, \ldots, n$$

form a basis of $\wedge\mathbf{p}$, and in fact

$$v_1 \cdots v_k \mapsto v_1 \wedge \cdots \wedge v_k \quad \text{if } (v_i, v_j) = 0 \ \forall i \neq j. \tag{9.1}$$

In particular, the map $C(\mathbf{p}) \to \wedge\mathbf{p}$ given above is an isomorphism of vector spaces, so we may identify $C(\mathbf{p})$ with $\wedge\mathbf{p}$ as a vector space if we choose, and then consider that $\wedge\mathbf{p}$ has two products: the Clifford product which we denote by juxtaposition and the exterior product which we denote with a $\wedge$.

Notice that this identification preserves the $\mathbf{Z}/2\mathbf{Z}$ gradation, an even element of the Clifford algebra is identified with an even element of the exterior algebra and an odd element is identified with an odd element.

9.1.5 The canonical antiautomorphism.

The Clifford algebra has a canonical anti-automorphism a which is the identity map on $\mathbf{p}$. In particular, for $v_i \in \mathbf{p}$ we have $a(v_1v_2) = v_2v_1$, $a(v_1v_2v_3) = v_3v_2v_1$, etc. By abuse of language, we use the same letter a to denote the similar anti-automorphism on $\wedge\mathbf{p}$ and observe from the above computations (in particular from the corresponding choice of bases) that a commutes with our identifying map $C(\mathbf{p}) \to \wedge\mathbf{p}$ so the notation is consistent. We have

$$a = (-1)^{\frac{1}{2}k(k-1)}\text{id} \quad \text{on } \wedge^k(\mathbf{p}).$$

For small values of k we have

k	$(-1)^{\frac{1}{2}k(k-1)}$
0	1
1	1
2	-1
3	-1
4	1
5	1
6	-1.

We will use subscripts to denote the homogeneous components of elements of $\wedge\mathbf{p}$. Notice that if $u \in \wedge^2\mathbf{p}$ then $au = -u$ by the above table, while $a(u^2) = (au)^2 = u^2$. Since u^2 is even (and hence has only even homogeneous components) and since the maximum degree of the homogeneous component of u^2 is 4, we conclude that

$$u^2 = (u^2)_0 + (u^2)_4 \quad \forall\ u \in \wedge^2\mathbf{p}. \tag{9.2}$$

For the same reason

$$v^2 = (v^2)_0 + (v^2)_4 \quad \forall\ v \in \wedge^3\mathbf{p}. \tag{9.3}$$

We also claim the following:

$$(ww')_0 = (aw, w') = (-1)^{\frac{1}{2}k(k-1)}(w, w') \quad \forall\ w, w' \in \wedge^k(\mathbf{p}). \tag{9.4}$$

Indeed, it is sufficient to verify this for w, w' belonging to a basis of $\wedge\mathbf{p}$, say the basis given by all elements of the form (9.1), in which case both sides of (9.4) vanish unless $w = w'$. If $w = w' = v_1 \wedge \cdots \wedge v_k$ (say) then

$$(ww)_0 = \iota(v_1)\cdots\iota(v_k)v_1 \wedge \cdots \wedge v_k =$$

$$(-1)^{\frac{1}{2}k(k-1)}(v_1, v_1)\cdots(v_k, v_k) = (-1)^{\frac{1}{2}k(k-1)}(w, w)$$

proving (9.4).

As special cases that we will use later on, observe that

$$(uu')_0 = -(u, u') \ \ \forall\ u, u' \in \wedge^2\mathbf{p} \tag{9.5}$$

and

$$(vv')_0 = -(v, v') \ \ \forall\ v, v' \in \wedge^3\mathbf{p}. \tag{9.6}$$

9.1.6 Commutator by an element of p.

For any $y \in \mathbf{p}$ consider the linear map

$$w \mapsto [y, w] = yw - (-1)^k wy \ \ \text{for } w \in \wedge^k\mathbf{p}$$

which is (anti)commutator in the Clifford multiplication by y. We claim that

$$[y, w] = 2\iota(y)w. \tag{9.7}$$

In particular, $[y, \cdot]$, which is automatically a derivation for the Clifford multiplication, is also a derivation for the exterior multiplication. Alternatively, this equation says that $\iota(y)$, which is a derivation for the exterior algebra multiplication, is also a derivation for the Clifford multiplication.

To prove (9.7) write

$$wy = a(ya(w)).$$

Then

$$yw = y \wedge w + \iota(y)w, \quad wy = a(y \wedge a(w)) + a(\iota(y)aw) = w \wedge y + (a\iota(y)a)w.$$

We may assume that $w \in \wedge^k \mathbf{p}$. Then

$$y \wedge w - (-1)^k w \wedge y = 0,$$

so we must show that

$$a\iota(y)aw = (-1)^{k-1}\iota(y)w.$$

For this we may assume that $y \neq 0$ and we may write

$$w = u \wedge z + z',$$

where $\iota(y)u = 1$ and $\iota(y)z = \iota(y)z' = 0$. In fact, we may assume that z and z' are sums of products of linear elements all of which are orthogonal to y. Then $\iota(y)az = \iota(y)az' = 0$ so

$$\iota(y)aw = (-1)^{k-1}az$$

since z has degree one less than w and hence

$$a\iota(y)aw = (-1)^{k-1}z = (-1)^{k-1}\iota(y)w. \quad QED$$

9.1.7 Commutator by an element of $\wedge^2\mathbf{p}$.

Suppose that

$$u \in \wedge^2\mathbf{p}.$$

Then for $y \in \mathbf{p}$ we have

$$[u, y] = -[y, u] = -2\iota(y)u. \tag{9.8}$$

In particular, if $u = y_i \wedge y_j$ where $y_i, y_j \in \mathbf{p}$ we have

$$[u, y] = 2(y_j, y)y_i - 2(y_i, y)y_j \quad \forall\ y \in \mathbf{p}. \tag{9.9}$$

If $(y_i, y_j) = 0$ this is an "infinitesimal rotation" in the plane spanned by y_i and y_j. Since $y_i \wedge y_j,\ i < j$ form a basis of $\wedge^2\mathbf{p}$ if $y_1, \ldots, y_n$ form an "orthonormal" basis of $\mathbf{p}$, we see that the map

$$u \mapsto [u, \cdot]$$

gives an isomorphism of $\wedge^2\mathbf{p}$ with the orthogonal algebra $o(\mathbf{p})$. This identification differs by a factor of two from the identification that we had been using earlier.

Now each element of $o(\mathbf{p})$ (in fact any linear transformation on $\mathbf{p}$) induces a derivation of $\wedge\mathbf{p}$. We claim that under the above identification of $\wedge^2\mathbf{p}$ with $o(\mathbf{p})$, the derivation corresponding to $u \in \wedge^2\mathbf{p}$ is Clifford commutation by u. In symbols, if θ_u denotes this induced derivation, we claim that

$$\theta_u(w) = [u, w] = uw - wu \quad \forall\ w \in \wedge\mathbf{p}. \tag{9.10}$$

To verify this, it is enough to check it on basis elements of the form (9.1), and hence by the derivation property for each v_j, where this reduces to (9.8).

We can now be more explicit about the degree four component of the Clifford square of an element of $\wedge^2\mathbf{p}$, i.e. the element $(u^2)_4$ occurring on the right of (9.2). We claim that for any three elements $y, y', y'' \ \in \mathbf{p}$

$$\frac{1}{2}\iota(y'')\iota(y')\iota(y)u^2 = (y\wedge y', u)\iota(y'')u + (y'\wedge y'', u)\iota(y)u + (y''\wedge y, u)\iota(y')u. \tag{9.11}$$

To prove this observe that

$$\begin{aligned}
\iota(y)u^2 &= (\iota(y)u)\,u + u\,(\iota(y)u) \\
\iota(y')\iota(y)u^2 &= (\iota(y')\iota(y)u)\,u - \iota(y)u\iota(y')u + \iota(y')u\iota(y)u + u\iota(y')\iota(y)u \\
&= 2\left((y\wedge y', u)u + \iota(y')u \wedge \iota(y)u\right) \\
\frac{1}{2}\iota(y'')\iota(y')\iota(y)u^2 &= (y\wedge y', u)\iota(y'')u + \iota(y'')\iota(y')u \wedge \iota(y)u - \iota(y')u \wedge \iota(y'')\iota(y)u \\
&= (y\wedge y', u)\iota(y'')u + (y'\wedge y'', u)\iota(y)u + (y''\wedge y, u)\iota(y')u
\end{aligned}$$

as required.

We can also be explicit about the degree zero component of u^2. Indeed, it follows from (9.9) that if $u = y_i \wedge y_j,\ i < j$ where $y_1, \ldots, y_n$ form an "orthonormal" basis of $\mathbf{p}$ then

$$\mathrm{tr}(\mathrm{ad}_{\mathbf{p}}\, u)^2 = -8(y_i, y_i)(y_j, y_j),$$

where $\mathrm{ad}_{\mathbf{p}}\, u$ denotes the (commutator) action of u on $\mathbf{p}$ under our identification of $\wedge^2\mathbf{p}$ with $o(\mathbf{p})$. But

$$(y_i \wedge y_j, y_i \wedge y_j) = (y_i, y_i)(y_j, y_j)(= \pm 1).$$

So using (9.5) we see that

$$(u^2)_0 = \frac{1}{8}\,\mathrm{tr}(\mathrm{ad}_{\mathbf{p}}\, u)^2 = -(u, u) \tag{9.12}$$

for $u \in \wedge^2\mathbf{p}$.

9.2 Orthogonal action of a Lie algebra.

Let $\mathbf{r}$ be a Lie algebra. Suppose that we have a representation of $\mathbf{r}$ acting as infinitesimal orthogonal transformations of $\mathbf{p}$ which means, in view of the identification of $\wedge^2\mathbf{p}$ with $o(\mathbf{p})$ that we have a map

$$\nu : \mathbf{r} \to \wedge^2\mathbf{p}$$

such that

$$x \cdot y = -2\iota(y)\nu(x) \tag{9.13}$$

where $x \cdot y$ denotes the action of $x \in \mathbf{r}$ on $y \in \mathbf{p}$.

9.2.1 Expression for ν in terms of dual bases.

It will be useful for us to write equation (9.13) in terms of a basis. So let $y_1, \ldots, y_n$ be a basis of $\mathbf{p}$ and let $z_1, \ldots, z_n$ be the dual basis relative to $(\ ,\ ) = (\ ,\ )_{\mathbf{p}}$. We claim that

$$\nu(x) = -\frac{1}{4}\sum_j y_j \wedge (x \cdot z_j). \tag{9.14}$$

Indeed, it suffices to verify (9.13) for each of the elements z_i. Now

$$\iota(z_i)\left(-\frac{1}{4}\sum_j y_j \wedge x \cdot z_j\right) = -\frac{1}{4}x \cdot z_i + \frac{1}{4}\sum_j (z_i, x \cdot z_j)y_j.$$

But

$$(z_i, x \cdot z_j) = -(x \cdot z_i, z_j)$$

since x acts as an infinitesimal orthogonal transformation relative to $(\ ,\ )$. So we can write the sum as

$$\frac{1}{4}\sum_j (z_i, x \cdot z_j)y_j = -\frac{1}{4}\sum_j (x \cdot z_i, z_j)y_j = -\frac{1}{4}x \cdot z_i$$

yielding

$$\iota(z_i)\left(-\frac{1}{4}\sum_j y_j \wedge x \cdot z_j\right) = -\frac{1}{2}x \cdot z_i$$

which is (9.13).

9.2.2 The adjoint action of a reductive Lie algebra.

For future use we record here a special case of (9.14): Suppose that $\mathbf{p} = \mathbf{r} = \mathbf{g}$ is a reductive Lie algebra with an invariant symmetric bilinear form, and the action is the adjoint action, i.e. $x \cdot y = [x, y]$. Let $\mathbf{h}$ be a Cartan subalgebra of $\mathbf{g}$

and let Φ denote the set of roots and suppose that we have chosen root vectors $e_\phi, e_{-\phi},\ \phi \in \Phi$ so that

$$(e_\phi, e_{-\phi}) = 1.$$

Let $h_1, \ldots, h_s$ be a basis of $\mathbf{h}$ and $k_1, \ldots k_s$ the dual basis. Let

$$\psi : \mathbf{g} \to \wedge^2 \mathbf{g}$$

be the map ν when applied to this adjoint action. Then (9.14) becomes

$$\psi(x) = \frac{1}{4}\left(\sum_{i=1}^{s} h_i \wedge [k_i, x] + \sum_{\phi\in\Phi} e_{-\phi} \wedge [e_\phi, x]\right). \tag{9.15}$$

In case $x = h \in \mathbf{h}$ this formula simplifies. The $[k_i, h] = 0$, and in the second sum we have

$$e_{-\phi} \wedge [e_\phi, h] = -\phi(h) e_{-\phi} \wedge e_\phi$$

which is invariant under the interchange of ϕ and $-\phi$. So let us make a choice Φ^+ of positive roots. Then we can write (9.15) as

$$\psi(h) = -\frac{1}{2}\sum_{\phi\in\Phi^+} \phi(h) e_{-\phi} \wedge e_\phi, \quad h \in \mathbf{h}. \tag{9.16}$$

Now

$$e_{-\phi} \wedge e_\phi = -1 + e_{-\phi} e_\phi.$$

So if

$$\rho := \frac{1}{2}\sum_{\phi\in\Phi^+} \phi \tag{9.17}$$

is one half the sum of the positive roots we have

$$\psi(h) = \rho(h) - \frac{1}{2}\sum_{\phi\in\Phi^+} \phi(h) e_{-\phi} e_\phi, \quad h \in \mathbf{h}. \tag{9.18}$$

In this equation, the multiplication on the right is in the Clifford algebra.

9.3 The spin representations.

If

$$\mathbf{p} = \mathbf{p}_1 \oplus \mathbf{p}_2$$

is a direct sum decomposition of a vector space $\mathbf{p}$ with a symmetric bilinear form into two orthogonal subspaces then it follows from the definition of the Clifford algebra that

$$C(\mathbf{p}) = C(\mathbf{p}_1) \otimes C(\mathbf{p}_2)$$

where the multiplication on the tensor product is taken in the sense of superalgebras, that is

$$(a_1 \otimes a_2)(b_1 \otimes b_2) := a_1 b_1 \otimes a_2 b_2$$

if either a_2 or b_1 are even, but

$$(a_1 \otimes a_2)(b_1 \otimes b_2) := -a_1 b_1 \otimes a_2 b_2$$

if both a_2 and b_1 are odd. It costs a sign to move one odd symbol past another.

9.3.1 The even dimensional case.

Suppose that $\mathbf{p}$ is even dimensional. If the metric is split (which is always the case if the metric is non-degenerate and we are over the complex numbers) then $\mathbf{p}$ is a direct sum of two dimensional mutually orthogonal split spaces, $\mathbf{W}_i$, so let us examine first the case of a two dimensional split space $\mathbf{p}$, spanned by ι, ϵ with $(\iota, \iota) = (\epsilon, \epsilon) = 0,\ (\iota, \epsilon) = \frac{1}{2}$. Let T be a one dimensional space with basis t and consider the linear map of $\mathbf{p} \to \mathrm{End}\ (\wedge T)$ determined by

$$\epsilon \mapsto \epsilon(t), \quad \iota \mapsto \iota(t^*)$$

where $\epsilon(t)$ denotes exterior multiplication by t and $\iota(t^*)$ denotes interior multiplication by t^*, the dual element to t in T^*. This is a Clifford map since

$$\epsilon(t)^2 = 0 = \iota(t^*)^2, \quad \epsilon(t)\iota(t^*) + \iota(t^*)\epsilon(t) = \mathrm{id}.$$

This therefore extends to a map of $C(\mathbf{p}) \to \mathrm{End}(\wedge T)$. Explicitly, if we use $1 \in \wedge^0 T,\ t \in \wedge^1 T$ as a basis of $\wedge T$ this map is given by

$$\begin{aligned}
1 &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\iota &\mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
\epsilon &\mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\
\iota\epsilon &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

This shows that the map is an isomorphism. If now

$$\mathbf{p} = \mathbf{W_1} \oplus \cdots \oplus \mathbf{W}_m$$

is a direct sum of two dimensional split spaces, and we write

$$T = T_1 \oplus \cdots \oplus T_m$$

where the $C(\mathbf{W}_i) \cong \mathrm{End}(\wedge T_i)$ as above, then since

$$\wedge T = \wedge T_1 \otimes \cdots \otimes \wedge T_m$$

we see that

$$C(\mathbf{p}) \cong \mathrm{End}\ (\wedge T).$$

In particular, $C(\mathbf{p})$ is isomorphic to the full $2^m \times 2^m$ matrix algebra and hence has a unique (up to isomorphism) irreducible module. One model of this is

$$S = \wedge T.$$

We can write

$$S = S_+ \oplus S_-$$

as a supervector space, where we choose the standard $\mathbf{Z}_2$ grading on $\wedge T$ to determine the grading on S if m is even, but use the opposite grading (for reasons which will become apparent in a moment) if m is odd.

The even part, $C_{\mathbf{0}}(\mathbf{p})$ of $C(\mathbf{p})$ acts irreducibly on each of $S_\pm$. Since $\wedge^2\mathbf{p}$ together with the constants generates $C_{\mathbf{0}}(\mathbf{p})$ we see that the action of $\wedge^2\mathbf{p}$ on each of $S_\pm$ is irreducible. Since $\wedge^2\mathbf{p}$ under Clifford commutation is isomorphic to $o(\mathbf{p})$ the two modules $S_\pm$ give irreducible modules for the even orthogonal algebra $o(\mathbf{p})$. These are the half spin representations of the even orthogonal algebras.

We can identify $S = S_+ \oplus S_-$ as a left ideal in $C(\mathbf{p})$ as follows: Suppose that we write

$$\mathbf{p} = \mathbf{p}_+ \oplus \mathbf{p}_-$$

where $\mathbf{p}_\pm$ are complementary isotropic subspaces. Choose a basis $e_1^+, \dots, e_m^+$ of $\mathbf{p}_+$ and let

$$e_+ := e_1^+ \cdots e_m^+ = e_1^+ \wedge \cdots \wedge e_m^+ \in \wedge^m \mathbf{p}_+.$$

We have

$$y_+ e_+ = 0, \quad \forall\ y_+ \in \mathbf{p}_+$$

and hence

$$(\wedge \mathbf{p}_+)_+ e_+ = 0.$$

In other words

$$\wedge \mathbf{p}_+ e_+$$

consists of all scalar multiples of e_+.

Since

$$\wedge \mathbf{p}_- \otimes \wedge \mathbf{p}_+ \to C(\mathbf{p}), \quad w_- \otimes w_+ \mapsto w_- w_+$$

is a linear bijection, we see that

$$C(\mathbf{p}) e_+ = \wedge \mathbf{p}_- e_+.$$

This means that the left ideal generated by e_+ in $C(\mathbf{p})$ has dimension 2^m, and hence must be isomorphic as a left $C(\mathbf{p})$ module to S. In particular it is a minimal left ideal.

Let $e_1^-, \dots e_m^-$ be a basis of $\mathbf{p}_-$ and for any subset $J = \{i_1, \dots, i_j\}, \quad i_1 < i_2 \cdots < i_j$ of $\{1, \dots, m\}$ let

$$e_-^J := e_{i_1}^- \wedge \cdots \wedge e_{i_j}^- = e_{i_1}^- \cdots e_{i_j}^-.$$

Then the elements

$$e_-^J e_+$$

form a basis of this model of S as J ranges over all subsets of $\{1, \dots, m\}$.

For example, suppose that we have a commutative Lie algebra $\mathbf{h}$ acting on $\mathbf{p}$ as infinitesimal isometries, so as to preserve each $\mathbf{p}_\pm$, that the e_i^+ are weight vectors corresponding to weights β_i and that the e_i^- form the dual basis, corresponding to the negative of these weights $-\beta_i$. Then it follows from (9.14) that the image, $\nu(h) \in \wedge^2(\mathbf{p}) \subset C(\mathbf{p})$ of an element $h \in \mathbf{h}$ is given by

$$\nu(h) = \frac{1}{2} \sum \beta_i(h) e_i^+ \wedge e_i^- = \frac{1}{2} \sum \beta_i(h)(1 - e_i^- e_i^+).$$

Thus

$$\nu(h) e_+ = \rho_{\mathbf{p}}(h) e_+ \tag{9.19}$$

where

$$\rho_{\mathbf{p}} := \frac{1}{2}(\beta_1 + \cdots + \beta_m). \tag{9.20}$$

For a subset J of $\{1, \dots, m\}$ let us set

$$\beta_J := \sum_{j \in J} \beta_j.$$

Then we have

$$[\nu(h), e_-^J] = -\beta_J(h) e_-^J$$

and so

$$\nu(h)(e_-^J e_+) = [\nu(h), e_-^J] e_+ + e_-^J \nu(h) e_+ = (\rho_{\mathbf{p}}(h) - \beta_J(h)) e_-^J e_+.$$

So if we denote the action of $\nu(h)$ on $S_\pm$ by $\mathrm{Spin}_\pm\, \nu(h)$ and the action of $\nu(h)$ on $S = S_+ \oplus S_-$ by $\mathrm{Spin}\, \nu(h)$ we have proved that

$$\text{The } e_-^J e_+ \text{ are weight vectors of Spin } \nu \text{ with weights } \rho_{\mathbf{p}} - \beta_J. \tag{9.21}$$

It follows from (9.21) that the difference of the characters of $\mathrm{Spin}_+\nu$ and $\mathrm{Spin}_-\nu$ is given by

$$\mathrm{ch}_{Spin_+\nu} - \mathrm{ch}_{Spin_-\nu} = \prod_j \left(e(\frac{1}{2}\beta_j) - e(-\frac{1}{2}\beta_j) \right) = e(\rho_{\mathbf{p}}) \prod_j \left(1 - e(-\beta_j)\right). \tag{9.22}$$

There are two special cases which are of particular importance: First, this applies to the case where we take $\mathbf{h}$ to be a Cartan subalgebra of $o(\mathbf{p}) = o(\mathbf{C}^{2k})$

itself, say the diagonal matrices in the block decomposition of $o(\mathbf{p})$ given by the decomposition

$$\mathbf{C}^{2k} = \mathbf{C}^k \oplus \mathbf{C}^k$$

into two isotropic subspaces. In this case the β_i is just the i-th diagonal entry and (9.22) yields the standard formula for the difference of the characters of the spin representations of the even orthogonal algebras.

A second very important case is where we take $\mathbf{h}$ to be the Cartan subalgebra of a semi-simple Lie algebra $\mathbf{g}$, and take

$$\mathbf{p} := \mathbf{n}_+ \oplus \mathbf{n}_-$$

relative to a choice of positive roots. Then the β_j are just the positive roots, and we see that the right hand side of (9.22) is just the Weyl denominator, the denominator occurring in the Weyl character formula. This means that we can write the Weyl character formula as

$$\mathrm{ch}(\mathrm{Irr}(\lambda) \otimes S_+) - \mathrm{ch}(\mathrm{Irr}(\lambda) \otimes S_-) = \sum_{w \in W} (-1)^w e(w \bullet \lambda)$$

where

$$w \bullet \lambda := w(\lambda + \rho).$$

If we let U_μ denote the one dimensional module for $\mathbf{h}$ given by the weight μ we can drop the characters from the preceding equation and simply write the Weyl character formula as an equation in virtual representations of $\mathbf{h}$:

$$\mathrm{Irr}(\lambda) \otimes S_+ - \mathrm{Irr}(\lambda) \otimes S_- = \sum_{w \in W} (-1)^w U_{w \bullet \lambda}. \tag{9.23}$$

The reader can now go back to the preceding chapter and to Theorem **??** where this version of the Weyl character formula has been generalized from the Cartan subalgebra to the case of a reductive subalgebra of equal rank. In the next chapter we shall see the meaning of this generalization in terms of the Kostant Dirac operator.

9.3.2 The odd dimensional case.

Since every odd dimensional space with a non-singular bilinear form can be written as a sum of a one dimensional space and an even dimensional space (both non-degenerate), we need only look at the Clifford algebra of a one dimensional space with a basis element x such that $(x, x) = 1$ (since we are over the complex numbers). This Clifford algebra is two dimensional, spanned by 1 and x with $x^2 = 1$, the element x being odd. This algebra clearly has itself as a canonical module under left multiplication and is irreducible as a $\mathbf{Z}/2\mathbf{Z}$ module. We may call this the spin representation of Clifford algebra of a one dimensional space. Under the even part of the Clifford algebra (i.e. under the scalars) it splits into two isomorphic (one dimensional) spaces corresponding to the basis $1, x$ of the Clifford algebra. Relative to this basis $1, x$ we have the left multiplication representation given by

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let us use $C(\mathbf{C})$ to denote the Clifford algebra of the one dimensional orthogonal vector space just described, and $S(\mathbf{C})$ its canonical module. Then if

$$\mathbf{q} = \mathbf{p} \oplus \mathbf{C}$$

is an orthogonal decomposition of an odd dimensional vector space into a direct sum of an even dimensional space and a one dimensional space (both non-degenerate) we have

$$C(\mathbf{q}) \cong C(\mathbf{p}) \otimes C(\mathbf{C}) \cong \mathrm{End}(S(\mathbf{q}))$$

where

$$S(\mathbf{q}) := S(\mathbf{p}) \otimes S(\mathbf{C})$$

all tensor products being taken in the sense of superalgebra. We have a decomposition

$$S(\mathbf{q}) = S_+(\mathbf{q}) \oplus S_-(\mathbf{q})$$

as a super vector space where

$$S_+(\mathbf{q}) = S_+(\mathbf{p}) \oplus xS_-(\mathbf{p}),\ S_-(\mathbf{q}) = S_-(\mathbf{p}) \oplus xS_+(\mathbf{p}).$$

These two spaces are equivalent and irreducible as $C_{\mathbf{0}}(\mathbf{q})$ modules. Since the even part of the Clifford algebra is generated by $\wedge^2\mathbf{q}$ together with the scalars, we see that either of these spaces is a model for the irreducible spin representation of $o(\mathbf{q})$ in this odd dimensional case.

Consider the decomposition $\mathbf{p} = \mathbf{p}_+ \oplus \mathbf{p}_-$ that we used to construct a model for $S(\mathbf{p})$ as being the left ideal in $C(\mathbf{p})$ generated by $\wedge^m \mathbf{p}_+$ where $m = \dim \mathbf{p}_+$. We have

$$\wedge(\mathbf{C} \oplus \mathbf{p}_-) = \wedge(\mathbf{C}) \otimes \wedge \mathbf{p}_-,$$

and

Proposition 1 *The left ideal in the Clifford algebra generated by $\wedge^m \mathbf{p}_+$ is a model for the spin representation.*

Notice that this description is valid for both the even and the odd dimensional case.

9.3.3 Spin ad and V_ρ.

We want to consider the following situation: $\mathbf{g}$ is a simple Lie algebra and we take $(\ ,\ )$ to be the Killing form. We have

$$\Phi : \mathbf{g} \to \wedge^2 \mathbf{g} \subset C(\mathbf{g})$$

which is the map ν associated to the adjoint representation of $\mathbf{g}$. Let $\mathbf{h}$ be a Cartan subalgebra and Φ the collection of roots. We choose root vectors $e_\phi, \phi \in \Phi$ so that

$$(e_\phi, e_{-\phi}) = 1.$$

Then it follows from (9.14) that

$$\Phi(x) = \frac{1}{4}\left(\sum h_i \wedge [k_i, x]_{\mathbf{g}} + \sum_{\phi\in\Phi} e_{-\phi} \wedge [e_\phi, x]_{\mathbf{g}}\right) \tag{9.24}$$

where the brackets are the Lie brackets of $\mathbf{g}$, where the h_i range over a basis of $\mathbf{h}$ and the k_i over a dual basis. This equation simplifies in the special cases where $x = h \in \mathbf{h}$ and in the case where $x = e_\psi,\ \psi \in \Phi^+$ relative to a choice, Φ^+ of positive roots. In the case that $x = h \in \mathbf{h}$ we have seen that $[k_i, h] = 0$ and the equation simplifies to

$$\Phi(h) = \rho(h)1 - \frac{1}{2}\sum_{\phi\in\Phi^+} \phi(h) e_{-\phi} e_\phi \tag{9.25}$$

where

$$\rho = \frac{1}{2}\sum_{\phi\in\Phi^+} \phi$$

is one half the sum of then positive roots.

We claim that for $\psi \in \Phi^+$ we have

$$\Phi(e_\psi) = \sum x_{\gamma'} e_{\psi'} \tag{9.26}$$

where the sum is over pairs (γ', ψ') such that either

1. $\gamma' = 0, \psi' = \psi$ and $x_{\gamma'} \in \mathbf{h}$ or
2. $\gamma' \in \Phi,\ \psi' \in \Phi_+$ and $\gamma' + \psi' = \psi,$ and $x_{\gamma'} \in \mathbf{g}_{\gamma'}$.

To see this, first observe that this first sum on the right of (9.24) gives

$$\sum \psi(k_i) h_i \wedge e_\psi$$

and so all these summands are of the form 1). For each summand

$$e_{-\phi} \wedge [e_\phi, e_\psi]$$

of the second sum, we may assume that either $\phi + \psi = 0$ or that $\phi + \psi \in \Phi$ for otherwise $[e_\phi, e_\psi] = 0$. If $\phi + \psi = 0$, so $\psi = -\phi \neq 0$, we have $[e_\phi, e_\psi] \in \mathbf{h}$ which is orthogonal to $e_{-\phi}$ since $\phi \neq 0$. So

$$e_{-\phi} \wedge [e_\phi, e_\psi] = -[e_\phi, e_\psi] e_\psi$$

again has the form 1).

If $\phi + \psi = \tau \neq 0$ is a root, then $(e_{-\phi}, e_\tau) = 0$ since $\phi \neq \tau$. If $\tau \in \Phi_+$ then

$$e_{-\phi} \wedge [e_\phi, e_\psi] = e_{-\phi} y_\tau,$$

where y_τ is a multiple of e_τ so this summand is of the form 2). If τ is a negative root, the ϕ must be a negative root so $-\phi$ is a positive root, and we can switch the order of the factors in the preceding expression at the expense of introducing a sign. So again this is of the form 2), completing the proof of (9.26).

Let $\mathbf{n}_+$ be the subalgebra of $\mathbf{g}$ generated by the positive root vectors and similarly $\mathbf{n}_-$ the subalgebra generated by the negative root vectors so

$$\mathbf{g} = \mathbf{n}_+ \oplus \mathbf{b}_-, \quad \mathbf{b}_- := \mathbf{n}_- \oplus \mathbf{h}$$

is an $\mathbf{h}$ stable decomposition of $\mathbf{g}$ into a direct sum of the nilradical and its opposite Borel subalgebra.

Let N be the number of positive roots and let

$$0 \neq n \in \wedge^N \mathbf{n}_+.$$

Clearly

$$yn = 0 \quad \forall y \in \mathbf{n}_+.$$

Hence by (9.26) we have

$$\Phi(\mathbf{n}_+)n = 0$$

while by (9.25)

$$\Phi(h)n = \rho(h)n \ \ \forall\ h \in \mathbf{h}.$$

This implies that the cyclic module

$$\Phi(U(\mathbf{g}))n$$

is a model for the irreducible representation V_ρ of $\mathbf{g}$ with highest weight ρ. Left multiplication by $\Phi(x), \ \ x \in \mathbf{g}$ gives the action of $\mathbf{g}$ on this module.

Furthermore, if $nc \neq 0$ for some $c \in C(\mathbf{g})$ then nc has the same property:

$$\Phi(\mathbf{n}_+)nc = 0, \quad \Phi(h)nc = \rho(h)nc, \ \ \forall\ h \in \mathbf{h}.$$

Thus every $nc \neq 0$ also generates a $\mathbf{g}$ module isomorphic to V_ρ.

Now the map

$$\wedge \mathbf{n}_+ \otimes \wedge \mathbf{b}_- \to C(\mathbf{g}), \quad x \otimes b \to xb$$

is a linear isomorphism and right Clifford multiplication of $\wedge^N \mathbf{n}_+$ by $\wedge \mathbf{n}_+$ is just $\wedge^N \mathbf{n}_+$, all the elements of of $\wedge^+ \mathbf{n}_+$ yielding 0. So we have the vector space isomorphism

$$nC(\mathbf{g}) = \wedge^N \mathbf{n}_+ \otimes \wedge \mathbf{b}_-.$$

In other words,

$$\Phi(U(\mathbf{g}))nC(\mathbf{g})$$

is a direct sum of irreducible modules all isomorphic to V_ρ with multiplicity equal to

$$\dim \wedge \mathbf{b}_- = 2^{s+N}$$

where $s = \dim \mathbf{h}$ and $N = \dim \mathbf{n}_- = \dim \mathbf{n}_+$. Let us compute the dimension of V_ρ using the Weyl dimension formula which asserts that for any irreducible finite dimensional representation V_λ with highest weight λ we have

$$\dim V_\lambda = \frac{\prod_{\phi \in \Phi_+} (\rho + \lambda, \phi)}{\prod_{\phi \in \Phi_+} (\rho, \phi)}.$$

If we plug in $\lambda = \rho$ we see that each factor in the numerator is twice the corresponding factor in the denominator so

$$\dim V_\rho = 2^N. \tag{9.27}$$

But then

$$\dim \Phi(U(\mathbf{g})) n C(\mathbf{g}) = 2^{s+2N} = \dim C(\mathbf{g}).$$

This implies that

$$C(\mathbf{g}) = \Phi(U(\mathbf{g})) n C(\mathbf{g}) = \Phi(U(\mathbf{g})) n (\wedge \mathbf{b}_-), \tag{9.28}$$

proving that $C(\mathbf{g})$ is primary of type V_ρ with multiplicity 2^{s+N} as a representation of $\mathbf{g}$ under the left multiplication action of $\Phi(\mathbf{g})$.

This implies that any submodule for this action, in particular any left ideal of $C(\mathbf{g})$, is primary of type V_ρ. Since we have realized the spin representation of $C(\mathbf{g})$ as a left ideal in $C(\mathbf{g})$ we have proved the important

Theorem 1 Spin ad *is primary of type* V_ρ.

One consequence of this theorem is the following:

Proposition 2 *The weights of* V_ρ *are*

$$\rho - \phi_J \tag{9.29}$$

where J *ranges over subsets of the positive roots and*

$$\phi_J = \sum_{\phi \in J} \phi_J$$

each occurring with multiplicity equal to the number of subsets J *yielding the same value of* ϕ_J.

Indeed, (9.21) gives the weights of Spin ad, but several of the β_J are equal due to the trivial action of ad($\mathbf{h}$) on itself. However this contribution to the multiplicity of each weight occurring in (9.21) is the same, and hence is equal to the multiplicity of V_ρ in Spin ad. So each weight vector of V_ρ must be of the form (9.29) each occurring with the multiplicity given in the proposition.