Math 128 Lecture 6

Complete reducibility for sl(2).
The classical algebras.

Review.

$sl(2)$ is spanned by the matrices:

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

They satisfy

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

For each non-negative integer n there is a unique irreducible representation of dimension n+1. We can find a basis so that this representation is given by

$$\rho_n(h) := \begin{pmatrix} n & 0 & \cdots & \cdots & 0 \\ 0 & n-2 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -n+2 & 0 \\ 0 & 0 & \cdots & \cdots & -n \end{pmatrix}, \; \rho_n(e) = \begin{pmatrix} 0 & n & \cdots & \cdots & 0 \\ 0 & 0 & n-1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix},$$

Review, continued.

$$\rho_n(f) := \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 1 & & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & n & 0 \end{pmatrix}.$$

The Casimir element C in U(sl(2)) is defined by

$$C := \frac{1}{2}h^2 + ef + fe \tag{5}$$

also

$$C = \frac{1}{2}h^2 + h + 2fe. \tag{6}$$

C lies in the **center** of the universal enveloping algebra of $sl(2)$, i.e. it commutes with all elements. If V is a module which possesses a "highest weight vector" v_λ as above, and if V has the property that v_λ is a cyclic vector, meaning that $V = U(L)v_\lambda$ then C takes on the constant value

$$C = \frac{\lambda(\lambda+2)}{2}\mathrm{Id}$$

Complete reducibility.

We will use the Casimir element C to prove that every finite dimensional representation W of $sl(2)$ is **completely reducible**, which means that if W' is an invariant subspace there exists a complementary invariant subspace W'' so that $W = W' \oplus W''$. Indeed we will prove:

Theorem 1 *1. Every finite dimensional representation of $sl(2)$ is completely reducible.*

2. Each irreducible subspace is a cyclic highest weight module with highest weight n where n is a non-negative integer.

3. When the representation is decomposed into a direct sum of irreducible components, the number of components with even highest weight is the multiplicity of 0 as an an eigenvector of h and

4. the number of components with odd highest weight is the multiplicity of 1 as an eigenvalue of h.

Proof. We know that every irreducible finite dimensional representation is a cyclic module with integer highest weight, that those with even highest weight contain 0 as an eigenvalue of h with multiplicity one and do not contain 1 as an eigenvalue of h, and that those with odd highest weight contain 1 as an eigenvalue of h with multiplicity one, and do not contain 0 as an eigenvalue. So 2), 3) and 4) follow from 1). We must prove 1).

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A special case.

Proposition 1 *Let* $0 \to V \to W \to k \to 0$ *be an exact sequence of* $sl(2)$ *modules and such that the action of* $sl(2)$ *on* k *is trivial (as it must be, since* $sl(2)$ *has no non-trivial one dimensional modules). Then this sequence splits, i.e. there is a line in* W *supplementary to* V *on which* $sl(2)$ *acts trivially.*

This proposition is, of course, a special case of the theorem we want to prove. But we shall see that it is sufficient to prove the theorem.

Reduction to the irreducible case.

Proof of proposition. It is enough to prove the proposition for the case that V is an irreducible module. Indeed, if V_1 is a submodule, then by induction on $\dim V$ we may assume the theorem is known for $0 \to V/V_1 \to W/V_1 \to k \to 0$ so that there is a one dimensional invariant subspace M in W/V_1 supplementary to V/V_1 on which the action is trivial. Let N be the inverse image of M in W. By another application of the proposition, this time to the sequence

$$0 \to V_1 \to N \to M \to 0$$

we find an invariant line, P, in N complementary to V_1. So $N = V_1 \oplus P$. Since $(W/V_1) = (V/V_1) \oplus M$ we must have $P \cap V = \{0\}$. But since $\dim W = \dim V + 1$, we must have $W = V \oplus P$. In other words P is a one dimensional subspace of W which is complementary to V.

Reduction to the faithful case.

Next we are reduced to proving the proposition for the case that $sl(2)$ acts faithfully on V. Indeed, let $I =$ the kernel of the action on V. Since $sl(2)$ is simple, either $I = sl(2)$ or $I = 0$. Suppose that $I = sl(2)$. For all $x \in sl(2)$ we have, by hypothesis, $xW \subset V$, and for $x \in I = sl(2)$ we have $xV = 0$. Hence

$$[sl(2), sl(2)] = sl(2)$$

acts trivially on all of W and the proposition is obvious. So we are reduced to the case that V is irreducible and the action, ρ, of $sl(2)$ on V is injective.

Using the Casimir.

We have our Casimir element C whose image in $\mathrm{End}\, W$ must map $W \to V$ since every element of $sl(2)$ does. On the other hand, $C = \frac{1}{2}n(n+2)$ Id $\neq 0$ since we are assuming that the action of $sl(2)$ on the irreducible module V is not trivial. In particular, the restriction of C to V is an isomorphism.

Hence ker $C_\rho : W \to V$ is an invariant line supplementary to V. We have proved the proposition.

Proving the theorem.

Proof of theorem from proposition. Let $0 \to E' \to E$ be an exact sequence of $sl(2)$ modules, and we may assume that $E' \neq 0$. We want to find an invariant complement to E' in E. Define W to be the subspace of $\mathrm{Hom}_k(E, E')$ whose restriction to E' is a scalar times the identity, and let $V \subset W$ be the subspace consisting of those linear transformations whose restrictions to E' is zero. Each of these is a submodule of $\mathrm{End}(E)$. We get a sequence

$$0 \to V \to W \to k \to 0$$

and hence a complementary line of invariant elements in W. In particular, we can find an element, T which is invariant, maps $E \to E'$, and whose restriction to E' is non-zero. Then $\ker T$ is an invariant complementary subspace. QED

The Weyl group.

We have

$$\exp e = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \exp -f = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

so

$$(\exp e)(\exp -f)(\exp e) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Since

$$\exp \operatorname{ad} x = Ad(\exp x)$$

we see that

$$\tau := (\exp \operatorname{ad} e)(\exp \operatorname{ad}(-f))(\exp \operatorname{ad} e)$$

consists of conjugation by the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Thus

$$\tau(h) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -h,$$

$$\tau(e) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = -f$$

and similarly $\tau(f) = -e$. In short

$$\tau : e \mapsto -f,\ f \mapsto -e,\ h \mapsto -h.$$

In particular, τ induces the "reflection" $h \mapsto -h$ on $\mathbf{C}h$ and hence the reflection $\mu \mapsto -\mu$ (which we shall also denote by s) on the (one dimensional) dual space. In any finite dimensional module V of $sl(2)$ the action of the element $\tau = (\exp e)(\exp -f)(\exp e)$ is defined, and

$$(\tau)^{-1}h(\tau) = \mathrm{Ad}\left(\tau^{-1}\right)(h) = s^{-1}h = sh$$

so if

$$hu = \mu u$$

then

$$h(\tau u) = \tau(\tau)^{-1}h(\tau)u = \tau s(h)u = -\mu\tau u = (s\mu)\tau u.$$

So if

$$V_\mu : \{u \in V | hu = \mu u\}$$

then

$$\tau(V_\mu) = V_{s\mu}. \tag{8}$$

The two element group consisting of the identity and the element s (acting as a reflection as above) is called the Weyl group of $sl(2)$. Its generalization to an arbitrary simple Lie algebra, together with the generalization of formula (8) will play a key role in what follows.

The classical simple algebras.

we introduce the "classical" finite dimensional simple Lie algebras, which come in four families: the algebras $sl(n+1)$ consisting of all traceless $(n+1) \times (n+1)$ matrices, the orthogonal algebras, on even and odd dimensional spaces (the structure for the even and odd cases are different) and the symplectic algebras (whose definition we will give below). We will prove that they are indeed simple by a uniform method - the method that we used in the preceding chapter to prove that $sl(2)$ is simple. So we axiomatize this method.

Graded simplicity.

We introduce the following conditions on the Lie algebra $\mathbf{g}$:

$$\mathbf{g} = \bigoplus_{i=-1}^{\infty} \mathbf{g}_i \tag{1}$$

$$[\mathbf{g}_i, \mathbf{g}_j] \subset \mathbf{g}_{i+j} \tag{2}$$

$$[\mathbf{g}_1, \mathbf{g}_{-1}] = \mathbf{g}_0 \tag{3}$$

$$[\mathbf{g}_{-1}, x] = 0 \Rightarrow x = 0, \ \forall\, x \in \mathbf{g}_i, \ \forall i \geq 0 \tag{4}$$

$$\text{There exists a } d \in \mathbf{g}_0 \quad \text{satisfying} \quad [d, x] = kx, \ x \in \mathbf{g}_k, \ \forall k, \tag{5}$$

and

$$\mathbf{g}_{-1} \ \text{ is irreducible under the (adjoint) action of } \mathbf{g}_0. \tag{6}$$

Condition (4) means that if $x \in \mathbf{g}_i, i \geq 0$ is such that $[y, x] = 0$ for all $y \in \mathbf{g}_{-1}$ then $x = 0$.

We wish to show that any non-zero $\mathbf{g}$ satisfying these six conditions is simple. We know that $\mathbf{g}_{-1}, \mathbf{g}_0$ and $\mathbf{g}_1$ are all non-zero, since $0 \neq d \in \mathbf{g}_0$ by (5) and $[\mathbf{g}_{-1}, \mathbf{g}_1] = \mathbf{g}_0$ by (3). So $\mathbf{g}$ can not be the one dimensional commutative algebra, and hence what we must show is that any non-zero ideal I of $\mathbf{g}$ must be all of $\mathbf{g}$.

Every ideal is graded.

We first show that any ideal I must be a **graded ideal**, i.e. that

$$I = I_{-1} \oplus I_0 \oplus I_1 \oplus, \quad \text{where } I_j := I \cap \mathbf{g}_j.$$

Indeed, write any $x \in \mathbf{g}$ as $x = x_{-1} + x_0 + x_1 + \cdots + x_k$ and successively bracket by d to obtain

$$\begin{aligned}
x &= x_{-1} + x_0 + x_1 + \cdots + x_k \\
[d, x] &= -x_{-1} + 0 + x_1 + \cdots + k x_k \\
[d, [d, x]] &= x_{-1} + 0 + x_1 + \cdots + k^2 x_k \\
\vdots \quad &\vdots \quad \vdots \\
(\operatorname{ad} d)^k x &= (-1)^k x_{-1} + 0 + x_1 + \cdots + k^k x_k \\
(\operatorname{ad} d)^{k+1} x &= (-1)^{k+1} x_{-1} + 0 + x_1 + \cdots + k^{k+1} x_k.
\end{aligned}$$

The matrix

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ -1 & 0 & 1 & \cdots & k \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ (-1)^k & 0 & 1 & \cdots & k^k \\ (-1)^{k+1} & 0 & 1 & \cdots & k^{k+1} \end{pmatrix}$$

is non singular. Indeed, it is a van der Monde matrix, that is a matrix of the form

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ t_1 & t_2 & 1 & \cdots & t_{k+2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ t_1^k & t_2^k & 1 & \cdots & t_{k+2}^k \\ t_1^{k+1} & t_2^{k+1} & 1 & \cdots & t_{k+2}^{k+1} \end{pmatrix}$$

whose determinant is

$$\prod_{i<j}(t_i - t_j)$$

and hence non-zero if all the t_j are distinct. Since $t_1 = -1, t_2 = 0, t_3 = 1$ etc. in our case, our matrix is invertible,

$$x = x_{-1} + x_0 + x_1 + \cdots + x_k$$

so we can
solve for each of the components of x in terms of the $(\mathrm{ad}\, d)^j x$. In particular, if $x \in I$ then all the $(\mathrm{ad}\, d)^j x \in I$ since I is an ideal, and hence all the component x_j of x belong to I as claimed.

The subspace $I_{-1} \subset \mathbf{g}_{-1}$ is invariant under the adjoint action of $\mathbf{g}_0$ on $\mathbf{g}_{-1}$, and since we are assuming that this action is irreducible, there are two possibilities: $I_{-1} = 0$ or $I_{-1} = \mathbf{g}_{-1}$. We will show that in the first case $I = 0$ and in the second case that $I = \mathbf{g}$.

$$[\mathbf{g}_{-1}, x] = 0 \qquad \Rightarrow \qquad x = 0, \ \forall \ x \in \mathbf{g}_i, \ \forall i \geq 0 \quad (4)$$

Condition (4) means that if $x \in \mathbf{g}_i, i \geq 0$ is such that $[y, x] = 0$ for all $y \in \mathbf{g}_{-1}$ then $x = 0$.

Indeed, if $I_{-1} = 0$ we will show inductively that $I_j = 0$ for all $j \geq 0$. Suppose $0 \neq y \in \mathbf{g}_0$. Since every element of $[I_{-1}, y]$ belongs to I and to $\mathbf{g}_{-1}$ we conclude that $[\mathbf{g}_{-1}, y] = 0$ and hence that $y = 0$ by (4). Thus $I_0 = 0$. Suppose that we know that $I_{j-1} = 0$. Then the same argument shows that any $y \in I_j$ satisfies $[\mathbf{g}_{-1}, y] = 0$ and hence $y = 0$. So $I_j = 0$ for all j, and since I is the sum of all the I_j we conclude that $I = 0$.

Now suppose that $I_{-1} = \mathbf{g}_{-1}$. Then $\mathbf{g}_0 = [\mathbf{g}_{-1}, \mathbf{g}_1] = [I_{-1}, \mathbf{g}_1] \subset I$. Furthermore, since $d \in \mathbf{g}_0 \subset I$ we conclude that $\mathbf{g}_k \subset I$ for all $k \neq 0$ since every element y of such a $\mathbf{g}_k$ can be written as $y = \frac{1}{k}[d, y] \in I$. Hence $I = \mathbf{g}$. QED

Examples.

In the homework you have verified that the Lie algebra of all polynomial vector fields is simple, as are the algebras sl(n) and the orthogonal algebras o(n) for n ≥ 5.

$o(4)$ is not simple, being isomorphic to $sl(2) \oplus sl(2)$: Indeed, if Z_1 and Z_2 are vector spaces equipped with non-degenerate anti-symmetric bilinear forms $\langle\ ,\rangle_1$ and $\langle\ ,\rangle_2$ then $Z_1 \otimes Z_2$ has a non-degenerate symmetric bilinear form $(\ ,\ )$ determined by

$$(u_1 \otimes u_2, v_1 \otimes v_2) = \langle u_1, v_1\rangle_1 \langle u_2, v_2\rangle_2.$$

The algebra $sl(2)$ acting on its basic two dimensional representation infinitesimally preserves the antisymmetric form given by

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle = x_1 y_2 - x_2 y_1.$$

Hence, if we take $Z = Z_1 = Z_2$ to be this two dimensional space, we see that $sl(2) \oplus sl(2)$ acts as infinitesimal orthogonal transformations on $Z \otimes Z$ which is four dimensional. But $o(4)$ is six dimensional so the embedding of $sl(2) \oplus sl(2)$ in $o(4)$ is in fact an isomorphism since $3 + 3 = 6$.

The symplectic algebras: 1- Poisson brackets.

We consider an even dimensional space with coordinates $q_1, q_2, \ldots, p_1, p_2, \ldots$. The polynomials have a Poisson bracket

$$\{f, g\} := \sum \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right). \tag{8}$$

This is clearly anti-symmetric, and direct computation will show that the Jacobi identity is satisfied. Here is a more interesting proof of Jacobi's identity: Notice that if f is a constant, then $\{f, g\} = 0$ for all g. So in doing bracket computations we can ignore constants. On the other hand, if we take g to be successively $q_1, \ldots, q_n, p_1, \ldots, p_n$ in (8) we see that the partial derivatives of f are completely determined by how it brackets with all g, in fact with all linear g. If we fix f, the map

$$h \mapsto \{f, h\}$$

is a **derivation**, i.e. it is linear and satisfies

$$\{f, h_1 h_2\} = \{f, h_1\} h_2 + h_1 \{f, h_2\}.$$

Proof of Jacobi's identity for the Poisson bracket.

$$h \mapsto \{f, h\}$$

is a **derivation**, i.e. it is linear and satisfies

$$\{f, h_1 h_2\} = \{f, h_1\} h_2 + h_1 \{f, h_2\}.$$

This follows immediately from from the definition (8). Now Jacobi's identity amounts to the assertion that

$$\{\{f, g\}, h\} = \{f, \{g, h\}\} - \{g, \{f, h\}\},$$

i.e. that the derivation

$$h \mapsto \{\{f, g\}, h\}$$

is the commutator of the of the derivations

$$h \mapsto \{f, h\} \quad \text{and} \qquad h \mapsto \{g, h\}.$$

Proof of Jacobi's identity for the Poisson bracket, continued.

We must show that the derivation $h \mapsto \{\{f,g\},h\}$

is the commutator of the of the derivations $h \mapsto \{f,h\}$ and $h \mapsto \{g,h\}$.
It is enough to check this on the

polynomials q_j and p_k. If we take $h = q_j$ then

$$\{f,q_j\} = \frac{\partial f}{\partial p_j}, \quad \{g,q_j\} = \frac{\partial g}{\partial p_j}$$

so

$$\begin{aligned}
\{f,\{g,q_j\}\} &= \sum\left(\frac{\partial f}{\partial p_i}\frac{\partial^2 g}{\partial q_i \partial p_j} - \frac{\partial f}{\partial q_i}\frac{\partial^2 g}{\partial p_i \partial p_j}\right) \\
\{f,\{f,q_j\}\} &= \sum\left(\frac{\partial g}{\partial p_i}\frac{\partial^2 f}{\partial q_i \partial p_j} - \frac{\partial g}{\partial q_i}\frac{\partial^2 f}{\partial p_i \partial p_j}\right) \text{ so} \\
\{f,\{g,q_j\}\} - \{g,\{f,q_j\}\} &= \frac{\partial}{\partial p_j}\{f,g\} \\
&= \{\{f,g\},q_j\}
\end{aligned}$$

as desired, with a similar computation for p_k.

Definition and simplicity of the symplectic algebras.

The symplectic algebra *sp(2n)* is defined to be the Lie subalgebra (of the Lie algebra of all polynomials under Poisson bracket) consisting of homogeneous quadratic polynomials. It is clear from the definition that the Poisson bracket of two homogenous quadratic polynomials is again a homogeneous quadratic polynomial. We make *sp(2n)* into a graded Lie algebra as follows:

Let $\mathbf{g}_1$ consist of homogeneous polynomials in the q's alone, so $\mathbf{g}_1$ is spanned by the $q_i q_j$.

Let $\mathbf{g}_{-1}$ be the quadratic polynomials in the p's alone, and let $\mathbf{g}_0$ be the mixed terms, so spanned by the $q_i p_j$. It is easy to see that $\mathbf{g}_0 \sim gl(n)$ and that $[\mathbf{g}_{-1}, \mathbf{g}_1] = \mathbf{g}_0$. To check that $\mathbf{g}_{-1}$ is irreducible under $\mathbf{g}_0$, observe that $[p_1 q_j, p_k p_\ell] = 0$ if $j \neq k$ or ℓ, and $[p_1 q_j, p_j p_\ell]$ is a multiple of $p_1 p_\ell$. So we can by one or two brackets carry any non-zero element of $\mathbf{g}_{-1}$ into a non-zero multiply of p_1^2, and then get any monomial from p_1^2 by bracketing with $p_i q_1$ appropriately. The element E is given by $\frac{1}{2}(p_1 q_1 + \cdots + p_n q_n)$.

Low dimensional coincidences.

We have already seen that $o(4) \sim sl(2) \oplus sl(2)$. We also have

$$o(6) \sim sl(4).$$

Both algebras are fifteen dimensional and both are simple. So to realize this isomorphism we need only find an orthogonal representation of $sl(4)$ on a six dimensional space. If we let $V = \mathbf{C}^4$ with the standard representation of $sl(4)$, we get a representation of $sl(4)$ on $\wedge^2(V)$ which is six dimensional. So we must describe a non-degenerate bilinear form on $\wedge^2 V$ which is invariant under the action of $sl(4)$. We

have a map, wedge product, of

$$\wedge^2 V \times \wedge^2 V \to \wedge^4 V.$$

Furthermore this map is symmetric, and invariant under the action of $gl(4)$. However $sl(4)$ preserves a basis (a non-zero element) of $\wedge^4 V$ and so we may identify $\wedge^4 V$ with $\mathbf{C}$. It is easy to check that the bilinear form so obtained is non-degenerate

We also have the identification

$$sp(4) \sim o(5)$$

both algebras being ten dimensional. To see this let $V = \mathbf{C}^4$ with an antisymmetric form ω preserved by $Sp(4)$. Then $\omega \otimes \omega$ induces a symmetric bilinear form on $V \otimes V$ as we have seen. Sitting inside $V \otimes V$ as an invariant subspace is $\wedge^2 V$ as we have seen, which is six dimensional. But $\wedge^2 V$ is not irreducible as a representation of $sp(4)$. Indeed, $\omega \in \wedge^2 V^*$ is invariant, and hence its kernel is a five dimensional subspace of $\wedge^2 V$ which is invariant under $sp(4)$. We thus get a non-zero homomorphism $sp(4) \to o(5)$ which must be an isomorphism since $sp(4)$ is simple.