Math 128 Lecture 7

The root structure of the classical algebras.

We are going to choose a basis for each of the classical simple algebras which generalizes the basis e, f, h that we chose for $sl(2)$. Indeed, for each classical simple algebra $\mathbf{g}$ we will first choose a maximal commutative subalgebra $\mathbf{h}$ all of whose elements are semi-simple = diagonizable in the adjoint representation. Since the adjoint action of all the elements of $\mathbf{h}$ commute, this means that they can be simultaneously diagonalized. Thus we can decompose $\mathbf{g}$ into a direct sum of simultaneous eigenspaces

$$\mathbf{g} = \mathbf{h} \oplus \bigoplus_{\alpha} \mathbf{g}_{\alpha} \tag{10}$$

where $0 \neq \alpha \in \mathbf{h}^*$ and

$$\mathbf{g}_{\alpha} := \{x \in \mathbf{g} | [h, x] = \alpha(h)x \ \forall \ h \in \mathbf{h}\}.$$

The linear functions α are called **roots** (originally because the $\alpha(h)$ are roots of the characteristic polynomial of $\mathrm{ad}(h)$). The simultaneous eigenspace $\mathbf{g}_{\alpha}$ is called the root space corresponding to α. The collection of all roots will usually be denoted by Φ.

$$A_n = sl(n+1).$$

We choose **h** to consist of the diagonal matrices in the algebra $sl(n+1)$ of all $(n+1)\times(n+1)$ matrices with trace zero. As a basis of **h** we take

$$\begin{aligned}
h_1 &:= \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \\
h_2 &:= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \\
\vdots &:= \vdots \\
h_n &:= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix}.
\end{aligned}$$

The roots of $A_n = sl(n+1)$.

Let L_i denote the linear function which assigns to each diagonal matrix its i-th (diagonal) entry,

Let E_{ij} denote the matrix with one in the i, j position and zero's elsewhere. Then

$$[h, E_{ij}] = (L_i(h) - L_j(h))E_{ij} \quad \forall \;\; h \in \mathbf{h}$$

so the linear functions of the form

$$L_i - L_j, \;\; i \neq j$$

are the roots.

Positive and negative roots.

We may subdivide the set of roots into two classes: the **positive roots**

$$\Phi^+ := \{L_i - L_j;\ i < j\}$$

and the **negative roots**

$$\Phi^- := -\Phi^+ = \{L_j - L_i,\ i < j\}.$$

Every root is either positive or negative. If we define

$$\alpha_i := L_i - L_{i+1}$$

then every positive root can be written as a sum of the α_i:

$$L_i - L_j = \alpha_i + \cdots + \alpha_{j-1}.$$

We have

$$\alpha_i(h_i) = 2,$$

and for $i \neq j$

$$\alpha_i(h_{i\pm 1}) = -1, \quad \alpha_i(h_j) = 0,\ j \neq i \pm 1. \tag{11}$$

Some *sl(2)* subalgebras.

The elements

$$E_{i,i+1},\ h_i,\ E_{i+1,i}$$

form a subalgebra of $sl(n+1)$ isomorphic to $sl(2)$. We may call it $sl(2)_i$.

The symplectic algebras $C_n = sp(2n), n \geq 2$.

Let $\mathbf{h}$ consist of all linear combinations of $p_1q_1, \ldots, p_nq_n$ and let L_i be defined by

$$L_i\left(a_1p_1q_1 + \cdots + a_np_nq_n\right) = a_i$$

so $L_1, \ldots, L_n$ is the basis of $\mathbf{h}^*$ dual to the basis $p_1q_1, \ldots, p_nq_n$ of $\mathbf{h}$.

The roots:

If $h = a_1p_1q_1 + \cdots + a_np_nq_n$ then

$$\begin{aligned} [h, q^iq^j] &= (a_i + a_j)q^iq^j \\ [h, q^ip^j] &= (a_i - a_j)q^ip^j \\ [h, p^ip^j] &= -(a_i + a_j)p^ip^j \end{aligned}$$

so the roots are

$$\pm(L_i + L_j) \text{ all } i, j \quad \text{and} \quad L_i - L_j \;\; i \neq j.$$

Positive and negative roots.

We can divide the roots Φ into positive and negative roots by setting

$$\Phi^+ = \{L_i + L_j\}_{\text{all } ij} \cup \{L_i - L_j\}_{i<j}.$$

If we set

$$\alpha_1 := L_1 - L_2, \ldots, \alpha_{n-1} := L_{n-1} - L_n, \alpha_n := 2L_n$$

then every positive root is a sum of the α_i. Indeed, $L_{n-1} + L_n = \alpha_{n-1} + \alpha_n$ and $2L_{n-1} = 2\alpha_{n-1} + \alpha_n$ and so on. In particular $2\alpha_{n-1} + \alpha_n$ is a root.

If we set

$$h_1 := p_1q_1 - p_2q_2, \ldots, h_{n-1} := p_{n-1}q_{n-1} - p_nq_n, \ h_n := p_nq_n$$

then

$$\alpha_i(h_i) = 2$$

while for $i \neq j$

$$\begin{aligned} \alpha_i(h_{i\pm 1}) &= -1, \quad i = 1, \ldots, n-1 \\ \alpha_i(h_j) &= 0, \ j \neq i \pm 1, i = 1, \ldots, n \\ \alpha_n(h_{n-1}) &= -2. \end{aligned} \tag{12}$$

Some *sl(2)* subalgebras of *sp(2n)*.

In particular, the elements $h_i, q_i p_{i+1}, q_{i+1} p_i$ for $i = 1, \ldots, n-1$ form a subalgebra isomorphic to $sl(2)$ as do the elements $h_n, \frac{1}{2}q_n^2, -\frac{1}{2}p_n^2$. We call these subalgebras $sl(2)_i,\ i = 1, \ldots, n$.

$D_n = o(2n), \; n \geq 3.$

We choose a basis $u_1, \ldots, u_n, v_1, \ldots, v_n$ of our orthogonal vector space V such that

$$(u_i, u_j) = (v_i, v_j) = 0, \forall\ i, j, \quad (u_i, v_j) = \delta_{ij}.$$

We let $\mathbf{h}$ be the subalgebra of $o(V)$ spanned by the $A_{u_i v_i}$, $i = 1, \ldots, n$. Here we have written A_{xy} instead of $A_{x \wedge y}$ in order to save space. We take

$$A_{u_1 v_1}, \ldots, A_{u_n v_n}$$

as a basis of $\mathbf{h}$ and let $L_1, \ldots, L_n$ be the dual basis.

The roots of *o(2n)* .

The bracket on o(2n) is given by

$$[A_{u\wedge v}, A_{x\wedge y}] = (v,x)A_{u\wedge y} - (u,x)A_{v\wedge y} - (v,y)A_{u\wedge x} + (u,y)A_{v\wedge x}. \quad (7)$$

Then

$$\pm L_k \pm L_\ell \; k \neq \ell$$

are the roots since from (7) we have

$$\begin{aligned}
[A_{u_i v_i}, A_{u_k u_\ell}] &= (\delta_{ik} + \delta_{i\ell})A_{u_k u_\ell} \\
[A_{u_i v_i}, A_{u_k v_\ell}] &= (\delta_{ik} - \delta_{i\ell})A_{u_k v_\ell} \\
[A_{u_i v_i}, A_{v_k v_\ell}] &= -(\delta_{ik} + \delta_{i\ell})A_{v_k v_\ell}.
\end{aligned}$$

The positive roots.

The roots are $\pm L_k \pm L_\ell \; k \neq \ell$

We can choose as positive roots the

$$L_k + L_\ell, L_k - L_\ell, \quad k < \ell$$

and set

$$\alpha_i := L_i - L_{i+1}, \; i = 1, \ldots, n-1, \quad \alpha_n := L_{n-1} + L_n.$$

Every positive root is a sum of these simple roots. If we set

$$h_i := A_{u_i v_i} - A_{u_{i+1} v_{i+1}}, i = 1, \ldots n-1,$$

and

$$h_n = A_{u_{n-1} v_{n-1}} + A_{u_n v_n}$$

then

$$\alpha_i(h_i) = 2$$

The simple roots.

These are

$$\alpha_i := L_i - L_{i+1},\ i = 1, \dots, n-1, \quad \alpha_n := L_{n-1} + L_n.$$

We set

$$h_i := A_{u_i v_i} - A_{u_{i+1} v_{i+1}}, i = 1, \dots n-1,$$

and

$$h_n = A_{u_{n-1} v_{n-1}} + A_{u_n v_n}$$

then

$$\alpha_i(h_i) = 2$$

and for $i \neq j$

$$\begin{aligned} \alpha_i(h_j) &= 0 \quad j \neq i \pm 1,\ i = 1, \dots n-2 \\ \alpha_i(h_{i\pm 1}) &= -1 \quad i = 1, \dots, n-2 \\ \alpha_{n-1}(h_{n-2}) &= -1 \\ \alpha_n(h_{n-2}) &= -1 \\ \alpha_n(h_{n-1}) &= 0. \end{aligned} \tag{13}$$

Subalgebras of *o(2n)* isomorphic to *sl(2)*.

For $i = 1, \ldots, n-1$ the elements $h_i, A_{u_i v_{i+1}}, A_{u_{i+1} v_i}$ form a subalgebra isomorphic to $sl(2)$ as do $h_n, A_{u_{n-1} u_n}, A_{v_{n-1} v_n}$.

$B_n = o(2n+1)$

We choose a basis $u_1, \ldots, u_n, v_1, \ldots, v_n, x$ of our orthogonal vector space V such that

$$(u_i, u_j) = (v_i, v_j) = 0, \forall\ i, j, \quad (u_i, v_j) = \delta_{ij},$$

and

$$(x, u_i) = (x, v_i) = 0\ \forall\ i, (x, x) = 1.$$

As in the even dimensional case we let $\mathbf{h}$ be the subalgebra of $o(V)$ spanned by the $A_{u_i v_i}$, $i = 1, \ldots, n$ and take

$$A_{u_1 v_1}, \ldots, A_{u_n v_n}$$

as a basis of $\mathbf{h}$ and let $L_1, \ldots, L_n$ be the dual basis. Then

$$\pm L_i \pm L_j\ i \neq j, \pm L_i$$

are roots.

The simple roots.

$$\pm L_i \pm L_j \;\; i \neq j, \pm L_i$$

are roots. We take

$$L_i \pm L_j, \; 1 \leq i < j \leq n, \text{ together with } L_i, \; i = 1, \ldots, n$$

to be the positive roots, and

$$\alpha_i := L_i - L_{i+1}, \; i = 1, \ldots, n-1, \; \alpha_n := L_n$$

to be the simple roots. We let

$$h_i := A_{u_i v_i} - A_{u_{i+1} v_{i+1}}, i = 1, \ldots n-1,$$

as in the even case, but set

$$h_n := 2A_{u_n v_n}.$$

Then every positive root can be written as a sum of the simple roots,

$$\alpha_i(h_i) = 2,\ i = 1, \dots n,$$

and for $i \neq j$

$$\begin{aligned} \alpha_i(h_j) &= 0 \ \ j \neq i \pm 1,\ i = 1, \dots n \\ \alpha_i(h_{i\pm 1}) &= -1 \ \ i = 1, \dots, n-2, n \\ \alpha_{n-1}(h_n) &= -2 \end{aligned} \tag{14}$$

Notice that in this case $\alpha_{n-1} + 2\alpha_n = L_{n-1} + L_n$ is a root. Finally we can construct subalgebras isomorphic to $sl(2)$, with the first $n-1$ as in the even orthogonal case and the last $sl(2)$ spanned by $h_n, A_{u_n x}, -A_{v_n x}$.

Diagrammatic presentation.

We can summarize the results obtained for each of the classical Lie algebras in the form of a diagram. The way to read these diagrams is as follows: Each node stands for a simple root with α_1 at the left. Two nodes α_i and α_j are connected by (one or more) edges if and only if $\alpha_i(h_j) \neq 0$.

For example, the diagram for A_ℓ is a simple chain with ℓ nodes:

This encodes the information that

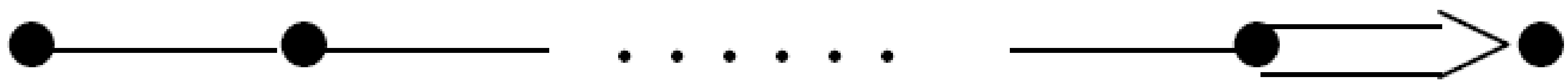
$$\alpha_i(h_i) = 2,$$

and for $i \neq j$

$$\alpha_i(h_{i\pm 1}) = -1, \quad \alpha_i(h_j) = 0, \ j \neq i \pm 1. \tag{11}$$

In all cases, the difference, $\alpha_i - \alpha_j$ is never a root, and, for $i \neq j$, $\alpha_i(h_j) \leq 0$ and is an integer. If, for $i \neq j$, $\alpha_i(h_j) < 0$ then $\alpha_i + \alpha_j$ is a root.

$B_\ell \; \ell \geq 3$



This encodes the information that

$$\alpha_i(h_i) = 2, \;\; i = 1, \ldots n,$$

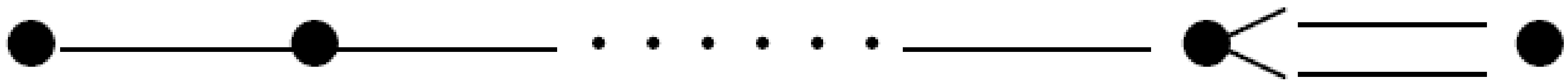
and for $i \neq j$

$$\begin{aligned} \alpha_i(h_j) &= 0 \;\; j \neq i \pm 1, \;\; i = 1, \ldots n \\ \alpha_i(h_{i\pm1}) &= -1 \;\; i = 1, \ldots, n-2, n \\ \alpha_{n-1}(h_n) &= -2 \end{aligned}$$

Notice that in this case $\alpha_{n-1} + 2\alpha_n = L_{n-1} + L_n$ is a root.

In two of the cases (B_ℓ and C_ℓ) it happens that $\alpha_i(h_j) = -2$. Then $\alpha_i + \alpha_j$ and $\alpha_i + 2\alpha_j$ are roots, and we draw a double bond with an arrow pointing towards α_j. In this case 2 is is the maximum integer such that $\alpha_i + k\alpha_j$ is a root. In all other cases, this maximum integer k is one if the nodes are connected (and zero it they are not).

$C_\ell \; \ell \geq 2$



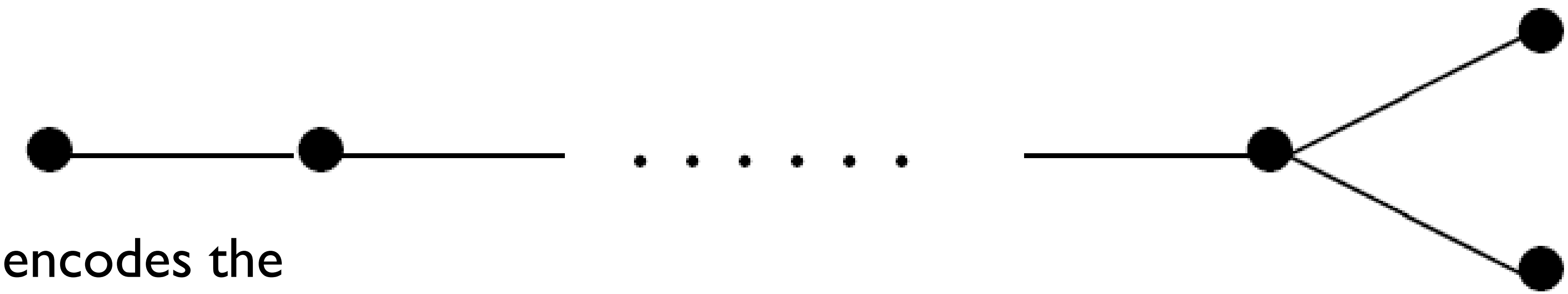
This encodes the information that

$$\alpha_i(h_i) = 2$$

while for $i \neq j$

$$\begin{aligned} \alpha_i(h_{i\pm 1}) &= -1, \quad i = 1, \ldots, n-1 \\ \alpha_i(h_j) &= 0, \; j \neq i \pm 1, i = 1, \ldots, n \\ \alpha_n(h_{n-1}) &= -2. \end{aligned}$$

$D_\ell \; \ell \geq 4$



This encodes the information that

$$\alpha_i(h_i) = 2$$

and for $i \neq j$

$$\begin{aligned}
\alpha_i(h_j) &= 0 \;\; j \neq i \pm 1,\; i = 1, \ldots n-2 \\
\alpha_i(h_{i\pm 1}) &= -1 \;\; i = 1, \ldots, n-2 \\
\alpha_{n-1}(h_{n-2}) &= -1 \\
\alpha_n(h_{n-2}) &= -1 \\
\alpha_n(h_{n-1}) &= 0. \\
\alpha_{n-1}(h_n) &= 0.
\end{aligned} \tag{13}$$

Extended diagrams.

$$\mathbf{g} = \mathbf{h} \oplus \bigoplus \mathbf{g}_\alpha \tag{10}$$

It follows from Jacobi's identity that in the decomposition (10), we have

$$[\mathbf{g}_\alpha, \mathbf{g}_{\alpha'}] \subset \mathbf{g}_{\alpha+\alpha'} \tag{15}$$

with the understanding that the right hand side is zero if $\alpha + \alpha'$ is not a root. In each of the cases examined above, every positive root is a linear combination of the simple roots with non-negative integer coefficients. Since the algebra is finite, there must be a **maximal** positive root β in the sense that $\beta+\alpha_i$ is not a root for any simple root. For example, in the case of $A_n = sl(n+1)$, the root $\beta := L_1 - L_{n+1}$ is maximal. The corresponding $\mathbf{g}_\beta$ consists of all $(n+1) \times (n+1)$ matrices with zeros everywhere except in the upper right hand corner. We can also consider the **minimal root** which is the negative of the maximal root, so

$$\alpha_0 := -\beta = L_{n+1} - L_1$$

in the case of A_n. Continuing to study this case, let

$$\alpha_0 := -\beta = L_{n+1} - L_1$$

in the case of A_n. Continuing to study this case, let

$$h_0 := h_{n+1} - h_1.$$

Then we have

$$\alpha_i(h_i) = 2, \ i = 0, \dots n$$

and

$$\alpha_0(h_1) = \alpha_0(h_n) = -1, \quad \alpha_0(h_i) = 0, \quad i \neq 0, 1, n.$$

This means that if we write out the $(n+1) \times (n+1)$ matrix whose entries are $\alpha_i(h_j), \ i, j = 0, \dots n$ we obtain a matrix of the form

$$2I - M$$

where $M_{ij} = 1$ if and only if $j = \pm 1$ with the understanding that $n + 1 = 0$ and $-1 = n$, i.e we do the subscript arithmetic mod n. In other words, M is the adjacency matrix of the cyclic graph with $n+1$ vertices labeled $0, \dots n$.

Also, we have

$$h_0 + h_1 + \cdots + h_n = 0.$$

If we apply α_i to this equation for $i = 0, \ldots n$ we obtain

$$(2I - M)\mathbf{1} = 0,$$

where $\mathbf{1}$ is the column vector all of whose entries are 1. We can write this equation as

$$M\mathbf{1} = \mathbf{2}\mathbf{1}.$$

In other words, $\mathbf{1}$ is an eigenvector of M with eigenvalue 2.

In the chapters that follow we shall see that any finite dimensional simple Lie algebra has roots, simple roots, maximal roots etc. giving rise to a matrix M with integer entries which is irreducible (in the sense of non-negative matrices - definition later on) and which has an eigenvector with positive (integer) entries with eigenvalue 2. This will allow us to classify the simple (finite dimensional) Lie algebras.