# Math 128 Lecture 16

The Weyl character formula and the Weyl dimension formula.

# Review: the value of the Casimir.

For any $\lambda \in \mathbf{h}^*$, any element $z \in Z(\mathbf{g})$ acts as a scalar, call it $\chi_\lambda(z)$ on the Verma module associated to $\lambda$.

In particular, if $\lambda$ is a dominant integral weight, it acts by this same scalar on the irreducible finite dimensional module associated to $\lambda$.

$$\chi_\lambda(z) = \lambda(\gamma(z)) \quad \forall\, z \in Z(\mathbf{g}). \tag{5}$$

We evaluated this formula on

$$\mathrm{Cas}^\kappa = \sum h_i k_i + \sum_\alpha x_\alpha z_\alpha$$

and found that

$$\chi_\lambda(\mathrm{Cas}^\kappa) = (\lambda + \rho, \lambda + \rho)_\kappa - (\rho, \rho)_\kappa$$

where

$$\rho := \frac{1}{2} \sum_{\alpha > 0} \alpha.$$

# The weight lattice.

We let $\mathbf{L} = \mathbf{L_g} \subset \mathbf{h}^*_{\mathbf{R}}$ denote the lattice of integral linear forms on $\mathbf{h}$, i.e.

$$\mathbf{L} = \{\mu \in \mathbf{h}^* | 2\frac{(\mu, \phi)}{(\phi, \phi)} \in \mathbf{Z}\ \forall \phi \in \Delta\}. \tag{9}$$

$\mathbf{L}$ is called the **weight lattice** of $\mathbf{g}$.

For $\mu, \lambda \in \mathbf{L}$ recall that

$$\mu \prec \lambda$$

if $\lambda - \mu$ is a sum of positive roots.

# Composition series for cyclic highest weight modules.

**Proposition 1** *Any cyclic highest weight module $Z(\lambda), \lambda \in \mathbf{L}$ has a composition series whose quotients are irreducible modules,* $\mathrm{Irr}(\mu)$ *where $\mu \prec \lambda$ satisfies*

$$(\mu + \rho, \mu + \rho)_\kappa = (\lambda + \rho, \lambda + \rho)_\kappa. \tag{10}$$

*In fact, if*

$$d = \sum \dim Z(\lambda)_\mu$$

*where the sum is over all $\mu$ satisfying (10) then there are at most $d$ steps in the composition series.*

**Remark.** There are only finitely many $\mu \in \mathbf{L}$ satisfying (10) since the set of all $\mu$ satisfying (10) is compact and $\mathbf{L}$ is discrete. Each weight is of finite multiplicity. Therefore $d$ is finite.

# Proof: the case d=1.

$$(\mu+\rho,\mu+\rho)_\kappa = (\lambda+\rho,\lambda+\rho)_\kappa. \qquad (10)$$

$$d = \sum \dim Z(\lambda)_\mu$$

**Proof by induction on $d$.** We first show that if $d = 1$ then $Z(\lambda)$ is irreducible. Indeed, if not, any proper submodule $W$, being the sum of its weight spaces, must have a highest weight vector with highest weight $\mu$, say. But then

$$\chi_\lambda(\mathrm{Cas}^\kappa) = \chi_\mu(\mathrm{Cas}^\kappa)$$

since $W$ is a submodule of $Z(\lambda)$ and $\mathrm{Cas}^\kappa$ takes on the constant value $\chi_\lambda(\mathrm{Cas}^\kappa)$ on $Z(\lambda)$. Thus $\mu$ and $\lambda$ both satisfy (10) contradicting the assumption $d = 1$.

# Proof by induction.

In general, suppose that $Z(\lambda)$ is not irreducible,

so has a submodule, $W$ and quotient module $Z(\lambda)/W$. Each of these is a cyclic highest weight module, and we have a corresponding composition series on each factor. In particular, $d = d_W + d_{Z(\lambda)/W}$ so that the $d$'s are strictly smaller for the submodule and the quotient module. Hence we can apply induction. QED

# Formal exponentials, characters.

For each $\lambda \in \mathbf{L}$ we introduce a formal symbol, $e(\lambda)$ which we want to think of as an "exponential" and so the symbols are multiplied according to the rule

$$e(\mu) \cdot e(\nu) = e(\mu + \nu). \qquad (11)$$

The *character* of a module $N$ is defined as

$$\mathrm{ch}_N = \sum \dim N_\mu \cdot e(\mu).$$

In all cases we will consider (cyclic highest weight modules and the like) all these dimensions will be finite, so the coefficients are well defined, but (in the case of Verma modules for example) there may be infinitely many terms in the (formal) sum. Logically, such a formal sum is nothing other than a function on $\mathbf{L}$ giving the "coefficient" of each $e(\mu)$.

In the case that $N$ is finite dimensional, the above sum is finite.

# Convolution.

If

$$f = \sum f_\mu e(\mu) \text{ and } g = \sum g_\nu e(\mu)$$

are two finite sums, then their product (using the rule (11)) corresponds to convolution:

$$\left(\sum f_\mu e(\mu)\right) \cdot \left(\sum g_\nu e(\nu)\right) = \sum (f \star g)_\lambda e(\lambda)$$

where

$$(f \star g)_\lambda := \sum_{\mu+\nu=\lambda} f_\mu g_\nu.$$

So we let $\mathbf{Z}_{\text{fin}}(\mathbf{L})$ denote the set of $\mathbf{Z}$ valued functions on $\mathbf{L}$ which vanish outside a finite set. It is a commutative ring under convolution, and we will oscillate in notation between writing an element of $\mathbf{Z}_{\text{fin}}(\mathbf{L})$ as an "exponential sum" thinking of it as a function of finite support.

# Characters and composition series.

Since we also want to consider infinite sums such as the characters of Verma modules, we enlarge the space $\mathbf{Z}_{\text{fin}}(\mathbf{L})$ by defining $\mathbf{Z}_{\text{gen}}(\mathbf{L})$ to consist of $\mathbf{Z}$ valued functions whose supports are contained in finite unions of sets of the form $\lambda - \sum_{\alpha \succ 0} k_\alpha \alpha$. The convolution of two functions belonging to $\mathbf{Z}_{\text{gen}}(\mathbf{L})$ is well defined, and belongs to $\mathbf{Z}_{\text{gen}}(\mathbf{L})$. So $\mathbf{Z}_{\text{gen}}(\mathbf{L})$ is again a ring.

It now follows from Prop.1 that

$$\text{ch}_{Z(\lambda)} = \sum \text{ch}_{\text{Irr}(\mu)}$$

where the sum is over the finitely many terms in the composition series.

# Irreducible characters.

$$(\mu+\rho,\mu+\rho)_\kappa = (\lambda+\rho,\lambda+\rho)_\kappa. \quad (10)$$

$$\mathrm{ch}_{Z(\lambda)} = \sum \mathrm{ch}_{\mathrm{Irr}(\mu)}$$

where the sum is over the finitely many terms in the composition series. In particular, we can apply this to $Z(\lambda) = \mathrm{Verm}(\lambda)$, the Verma module. Let us order the $\mu_i \prec \lambda$ satisfying (10) in such a way that $\mu_i \prec \mu_j \Rightarrow i \leq j$. Then for each of the finitely many $\mu_i$ occurring we get a corresponding formula for $\mathrm{ch}_{\mathrm{Verm}(\mu_i)}$ and so we get collection of equations

$$\mathrm{ch}_{\mathrm{Verm}(\mu_j)} = \sum a_{ij} \mathrm{ch}_{\ \mathrm{Irr}(\mu_i)}$$

where $a_{ii} = 1$ and $i \leq j$ in the sum. We can invert this upper triangular matrix and therefore conclude that there is a formula of the form

$$\mathrm{ch}_{\mathrm{Irr}(\lambda)} = \sum b(\mu) \mathrm{ch}_{\mathrm{Verm}(\mu)} \quad (12)$$

where the sum is over $\mu \prec \lambda$ satisfying (10) with coefficients $b(\mu)$ that we shall soon determine. But we do know that $b(\lambda) = 1$.

# The Weyl character formula.

$$\mathrm{ch}_{\mathrm{Irr}(\lambda)} = \sum b(\mu)\mathrm{ch}_{\mathrm{Verm}(\mu)} \tag{12}$$

**Proposition 2** *The non-zero coefficients in (12) occur only when*

$$\mu = w(\lambda + \rho) - \rho$$

*where $w \in W$, the Weyl group of $\mathbf{g}$, and then*

$$b(\mu) = (-1)^w.$$

*Here*

$$(-1)^w := \det w.$$

We will prove this by proving some combinatorial facts about multiplication of sums of exponentials.

# Notational recall.

We recall our notation: For $\lambda \in \mathbf{h}^*$, $\mathrm{Irr}(\lambda)$ denotes the unique irreducible module of highest weight, $\lambda$,and $\mathrm{Verm}(\lambda)$ denotes the Verma module of highest weight $\lambda$, and more generally, $Z(\lambda)$ denotes an arbitrary cyclic module of highest weight $\lambda$. Also

$$\rho := \frac{1}{2} \sum_{\phi \in \Phi^+} \phi$$

is one half the sum of the positive roots. Let $\lambda_i, i = 1, \ldots, \dim \mathbf{h}$ be the basis of the weight lattice, $L$ dual to the base $\Delta$. So

$$\lambda_i(h_{\alpha_j}) = \langle \lambda_i, \alpha_j \rangle = \delta_{ij}.$$

# An alternative expression for $\rho$ .

$$\lambda_i(h_{\alpha_j}) = \langle \lambda_i, \alpha_j \rangle = \delta_{ij}.$$

Since $s_i(\alpha_i) = -\alpha_i$ while keeping all the other positive roots positive, we saw that this implied that

$$s_i\rho = \rho - \alpha_i$$

and therefore

$$\langle \rho, \alpha_i \rangle = 1, \quad i = 1, \ldots, \ell := \dim(\mathbf{h}).$$

In other words

$$\rho = \frac{1}{2} \sum_{\phi \in \Phi^+} \phi = \lambda_1 + \cdots + \lambda_\ell. \tag{13}$$

# The Kostant partition function.

The **Kostant partition function**, $P_K(\mu)$ is defined as the number of sets of non-negative integers, $k_\beta$ such that

$$\mu = \sum_{\beta \in \Phi^+} k_\beta \beta.$$

(The value is zero if $\mu$ can not be expressed as a sum of positive roots.)

For any module $N$ and any $\mu \in \mathbf{h}^*$, $N_\mu$ denotes the weight space of weight $\mu$. For example, in the Verma module, $\mathrm{Verm}(\lambda)$, the only non-zero weight spaces are the ones where $\mu = \lambda - \sum_{\beta \in \Phi^+} k_\beta \beta$ and the multiplicity of this weight space, i.e. the dimension of $\mathrm{Verm}(\lambda)_\mu$ is the number of ways of expressing in this fashion, i.e.

$$\dim \ \mathrm{Verm}(\lambda)_\mu = P_K(\lambda - \mu). \tag{14}$$

# The character of a Verma module.

$$\dim \ \mathrm{Verm}(\lambda)_\mu = P_K(\lambda - \mu). \tag{14}$$

In terms of the character notation introduced in the preceding section we can write this as

$$\mathrm{ch}_{\mathrm{Verm}(\lambda)} = \sum P_K(\lambda - \mu) e(\mu).$$

To be consistent with Humphreys' notation, define the *Kostant function p* by

$$p(\nu) = P_K(-\nu)$$

and then in succinct language

$$\mathrm{ch}_{\mathrm{Verm}(\lambda)} = p(\cdot - \lambda). \tag{15}$$

# The characterof a Verma module, 2.

$$\mathrm{ch}_{\mathrm{Verm}(\lambda)} = p(\cdot - \lambda). \tag{15}$$

Observe that if

$$f = \sum f(\mu)e(\mu)$$

then

$$f \cdot e(\lambda) = \sum f(\mu)e(\lambda + \mu) = \sum f(\nu - \lambda)e(\nu).$$

We can express this by saying that

$$f \cdot e(\lambda) = f(\cdot - \lambda).$$

Thus, for example,

$$\mathrm{ch}_{\mathrm{Verm}(\lambda)} = p(\cdot - \lambda) = p \cdot e(\lambda).$$

# The geometric series.

$$\mathrm{ch}_{\mathrm{Verm}(\lambda)} = p(\cdot - \lambda) = p \cdot e(\lambda).$$

Also observe that if

$$f_\alpha = \frac{1}{1 - e(-\alpha)} := 1 + e(-\alpha) + e(-2\alpha) + \cdots$$

then

$$(1 - e(-\alpha)) f_\alpha = 1$$

and

$$\prod_{\alpha \in \Phi^+} f_\alpha = p$$

by the definition of the Kostant function.

# The Weyl denominator.

Define the function $q$ by

$$q := \prod_{\alpha \in \Phi^+} (e(\alpha/2) - e(-\alpha/2)) = e(\rho) \prod (1 - e(-\alpha))$$

since $e(\rho) = \prod_{\alpha \in \Phi^+} e(\alpha/2)$. Notice that

$$wq = (-1)^w q.$$

It is enough to check this on fundamental reflections, but they have the property that they make exactly one positive root negative, hence change the sign of $q$.

# The Weyl denominator and the character of a Verma module.

We have

$$qp = e(\rho). \tag{16}$$

Indeed,

$$\begin{aligned} qpe(-\rho) &= \left[\prod(1-e(-\alpha))\right]e(\rho)pe(-\rho) \\ &= \left[\prod(1-e(-\alpha))\right]p \\ &= \prod(1-e(-\alpha))\prod f_\alpha \\ &= 1. \end{aligned}$$

Therefore,

$$q\mathrm{ch}_{\mathrm{Verm}(\lambda)} = qpe(\lambda) = e(\rho)e(\lambda) = e(\lambda+\rho).$$

# Proof of the **WCF**, 1 .

We know that

$$\mathrm{ch}_{\mathrm{Irr}(\lambda)} = \sum b(\mu)\mathrm{ch}_{\mathrm{Verm}(\mu)} \tag{12}$$

where the sum is over $\mu \prec \lambda$ satisfying

$$(\mu + \rho, \mu + \rho)_\kappa = (\lambda + \rho, \lambda + \rho)_\kappa. \tag{10}$$

and

$$q\mathrm{ch}_{\mathrm{Verm}(\lambda)} = qpe(\lambda) = e(\rho)e(\lambda) = e(\lambda + \rho).$$

Let us now multiply both sides of (12) by $q$ and use the preceding equation. We obtain

$$q\mathrm{ch}_{\mathrm{Irr}(\lambda)} = \sum b(\mu)e(\mu + \rho)$$

where the sum is over all $\mu \prec \lambda$ satisfying (10), and the $b(\mu)$ are coefficients we must determine.

# Proof of the **WCF**, 2 .

$$q\mathrm{ch}_{\mathrm{Irr}(\lambda)} = \sum b(\mu)e(\mu+\rho)$$

Now ch $_{\mathrm{Irr}(\lambda)}$ is invariant under the Weyl group $W$, and $q$ transforms by $(-1)^w$. Hence if we apply $w \in W$ to the preceding equation we obtain

$$(-1)^w q\mathrm{ch}_{\mathrm{Irr}(\lambda)} = \sum b(\mu)e(w(\mu+\rho)).$$

This shows that the set of $\mu+\rho$ with non-zero coefficients is stable under $W$ and the coefficients transform by the sign representation for each $W$ orbit. In particular, each element of the form $\mu = w(\lambda+\rho)-\rho$ has $(-1)^w$ as its coefficient. We can thus write

$$q\mathrm{ch}_{V(\lambda)} = \sum_{w\in W} (-1)^w e(w(\lambda+\rho)) + R$$

where $R$ is a sum of terms corresponding to $\mu+\rho$ which are not of the form $w(\lambda+\rho)$.

# Proof of the **WCF**, 3 .

$$q\mathrm{ch}_{V(\lambda)} = \sum_{w\in W} (-1)^w e(w(\lambda+\rho)) + R$$

where $R$ is a sum of terms corresponding to $\mu+\rho$ which are not of the form $w(\lambda+\rho)$. We claim that there are no such terms and hence $R=0$. Indeed, if there were such a term, the transformation properties under $W$ would demand that there be such a term with $\mu+\rho$ in the closure of the Weyl chamber, i.e.

$$\mu+\rho \in \Lambda := \mathbf{L} \cap D$$

where

$$D = D_{\mathbf{g}} = \{\lambda \in E | (\lambda, \phi) \geq 0 \ \ \forall \phi \in \Delta^+\}$$

and $E = \mathbf{h}^*_{\mathbf{R}}$ denotes the space of real linear combinations of the roots. But we claim that

$$\mu \prec \lambda, \ \ (\mu+\rho, \mu+\rho) = (\lambda+\rho, \lambda+\rho), \ \ \& \ \mu+\rho \in \Lambda \implies \mu = \lambda.$$

# Proof of the **WCF**, 4 .

$$\mu \prec \lambda, \quad (\mu+\rho, \mu+\rho) = (\lambda+\rho, \lambda+\rho), \quad \& \; \mu+\rho \in \Lambda \implies \mu = \lambda.$$

Indeed, write $\mu = \lambda - \pi, \quad \pi = \sum k_\alpha \alpha, \; k_\alpha \geq 0$ so

$$\begin{aligned} 0 &= (\lambda+\rho, \lambda+\rho) - (\mu+\rho, \mu+\rho) \\ &= (\lambda+\rho, \lambda+\rho) - (\lambda+\rho-\pi, \lambda+\rho-\pi) \\ &= (\lambda+\rho, \pi) + (\pi, \mu+\rho) \\ &\geq (\lambda+\rho, \pi) \;\; \text{since } \mu+\rho \in \Lambda \\ &\geq 0 \end{aligned}$$

since $\lambda + \rho \in \Lambda$ and in fact lies in the interior of $D$. But the last inequality is strict unless $\pi = 0$. Hence $\pi = 0$.

we have derived the fundamental formula

$$q\mathrm{ch}_{\mathrm{Irr}(\lambda)} = \sum_{w \in W} (-1)^w e(w(\lambda+\rho)). \tag{17}$$

# Completion of the proof of the **WCF**.

$$q\mathrm{ch}_{\mathrm{Irr}(\lambda)} = \sum_{w \in W} (-1)^w e(w(\lambda + \rho)). \tag{17}$$

Notice that if we take $\lambda = 0$ and so the trivial representation with character 1 for $V(\lambda)$, (17) becomes

$$q = \sum (-1)^w e(w\rho)$$

and this is precisely the denominator in the **Weyl character formula**:

$$\textbf{WCF} \ \mathrm{ch}_{\mathrm{Irr}(\lambda)} = \frac{\sum_{w \in W} (-1)^w e(w(\lambda + \rho))}{\sum_{w \in W} (-1)^w e(w\rho)} \tag{18}$$

# The Weyl numerator and the Weyl denominator.

For any weight, $\mu$ we define

$$A_\mu := \sum_{w \in W} (-1)^w e(w\mu).$$

Then we can write the Weyl character formula as

$$\mathrm{ch}_{\mathrm{Irr}(\lambda)} = \frac{A_{\lambda+\rho}}{A_\rho}.$$

# Preliminaries to the Weyl dimension formula.

For any weight $\mu$ define the homomorphism $\Psi_\mu$ from the ring $\mathbf{Z}_{\text{fin}}(\mathbf{L})$ into the ring of formal power series in one variable $t$ by the formula

$$\Psi_\mu(e(\nu)) = e^{(\nu,\mu)_\kappa t}$$

(and extend linearly). The left hand side of the Weyl character formula belongs to $\mathbf{Z}_{\text{fin}}(\mathbf{L})$, and hence so does the right hand side which is a quotient of two elements of $\mathbf{Z}_{\text{fin}}(\mathbf{L})$. Therefore for any $\mu$ we have

$$\Psi_\mu(\text{ch}_{\text{Irr}(\lambda)}) = \frac{\Psi_\mu(A_{\rho+\lambda})}{\Psi_\mu(A_\rho)}.$$

We claim that

$$\Psi_\mu(A_\nu) = \Psi_\nu(A_\mu)$$

for any pair of weights.

# More preliminaries to the **WDF**.

$$\Psi_\mu(A_\nu) = \Psi_\nu(A_\mu)$$

for any pair of weights. Indeed,

$$\begin{aligned}
\Psi_\mu(A_\nu) &= \sum_w (-1)^w e^{(\mu, w\nu)_\kappa t} \\
&= \sum_w (-1)^w e^{(w^{-1}\mu, \nu)_\kappa t} \\
&= \sum (-1)^w e^{(w\mu, \nu)_\kappa t} \\
&= \Psi_\nu(A_\mu).
\end{aligned}$$

# More preliminaries to the **WDF**.

We have proved that $\Psi_\mu(A_\nu) = \Psi_\nu(A_\mu)$

so

$$\begin{aligned}
\Psi_\rho(A_\lambda) &= \Psi_\lambda(A_\rho) \\
&= \Psi_\lambda(q) \\
&= \Psi_\lambda\left(\prod(e(\alpha/2) - e(-\alpha/2))\right) \\
&= \prod_{\alpha\in\Phi^+}\left(e^{(\lambda,\alpha)_\kappa t/2} - e^{-(\lambda,\alpha)_\kappa t/2)}\right) \\
&= \left(\prod(\lambda,\alpha)_\kappa\right) t^{\#\Phi^+} + \text{ terms of higher degree in } t.
\end{aligned}$$

Hence

$$\Psi_\rho(\mathrm{ch}_{\mathrm{Irr}(\lambda)}) = \frac{\Psi_\rho(A_{\lambda+\rho})}{\Psi_\rho(A_\rho)} = \frac{\prod(\lambda+\rho,\alpha)_\kappa}{\prod(\rho,\alpha)_\kappa} + \text{ terms of positive degree in } t.$$

# The Weyl dimension formula.

$$\Psi_\rho(\mathrm{ch}_{\mathrm{Irr}(\lambda)}) = \frac{\Psi_\rho(A_{\lambda+\rho})}{\Psi_\rho(A_\rho)} = \frac{\prod(\lambda+\rho,\alpha)_\kappa}{\prod(\rho,\alpha)_\kappa} + \text{ terms of positive degree in } t.$$

Now consider the composite homomorphism: first apply $\Psi_\rho$ and then set $t = 0$. This has the effect of replacing every $e(\mu)$ by the constant 1. Hence applied to the left hand side of the Weyl character formula this gives the dimension of the representation $\mathrm{Irr}(\lambda)$. The previous equation shows that when this composite homomorphism is applied to the right hand side of the Weyl character formula, we get the right hand side of the **Weyl dimension formula**:

$$\dim \mathrm{Irr}(\lambda) = \frac{\prod_{\alpha\in\Phi^+}(\lambda+\rho,\alpha)_\kappa}{\prod_{\alpha\in\Phi^+}(\rho,\alpha)_\kappa}. \tag{19}$$