Math 129: Algebraic Number Theory
Homework Assignment 2

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Due: Thursday, February 26, 2004

The Problems:

1. (a) Let k be a field. Prove that $k[x]$ is a Dedekind domain.
   (b) (Problem 1.12 from Swinnerton-Dyer) Let x be an indeterminate. Show that the ring $\mathbf{Z}[x]$ is Noetherian and integrally closed in its field of fractions, but is not a Dedekind domain.

2. Use MAGMA to write each of the following (fractional) ideals as a product of explicitly given prime ideals:
   (a) The ideal (2004) in $\mathbf{Q}(\sqrt{-1})$.
   (b) The ideals $I = (7)$ and $J = (3)$ in the ring of integers of $\mathbf{Q}(\zeta_7)$, where ζ_7 is a root of the irreducible polynomial $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$. (The field $\mathbf{Q}(\zeta_7)$ is called the 7th cyclotomic field.)
   (c) The principal fractional ideal $(3/8)$ in $\mathbf{Q}(\sqrt{5})$.

3. Suppose R is an order in the ring $\mathcal{O}_K$ of integers of a number field. (Recall that an order is a subring of finite index in $\mathcal{O}_K$.) For each of the following questions, either explain why the answer is yes for any possible order R in any $\mathcal{O}_K$, or find one specific counterexample:
   (a) Is R necessarily Noetherian?
   (b) Is R necessarily integrally closed in its field of fractions?
   (c) Is every nonzero prime ideal of R necessarily maximal?
   (d) Is it always possible to write every ideal of R uniquely as a product of prime ideals of R?

4. Let $\mathcal{O}_K$ be the ring of integers of a number field K. Prove that the group of fractional ideals of $\mathcal{O}_K$, under multiplication is (non-canonically) isomorphic to the group of positive rational numbers under multiplication.