**Homework 2:** VALUATIONS AND ABSOLUTE VALUES II

Questions marked with an * are optional, i.e. not for credit.

1) Find a rational number $\alpha$ such that

$$|\alpha|_{\mathbf{R}} < 1/10; \quad |\alpha - 1|_2 < 1/10; \quad \text{and } |\alpha + 1|_3 < 1/10.$$

2) Find $f \in \mathbf{Q}(T)$ such that

$$\mathrm{ord}_0(f - 1) > 2 \quad \text{and} \quad \mathrm{ord}_1(f + 1) > 2.$$

3) Consider the absolute values $|\ |_1$ (resp. $|\ |_2$) on $\mathbf{Q}(\sqrt[3]{17})$ obtained by composing $|\ |_{\mathbf{R}}$ with the embedding

$$\mathbf{Q}(\sqrt[3]{17}) \hookrightarrow \mathbf{R}$$

sending $\sqrt[3]{17}$ to the unique real cube root of 17 (resp. obtained by composing $|\ |_{\mathbf{C}}^{1/2}$ with the embedding

$$\mathbf{Q}(\sqrt[3]{17}) \hookrightarrow \mathbf{C}$$

sending $\sqrt[3]{17}$ to the unique cube root of 17 with positive imaginary part). Find $\alpha \in \mathbf{Q}(\sqrt[3]{17})$ with

$$|\alpha - 1|_1 < 1/10$$

and

$$|\alpha + 1|_2 < 1/10.$$

4) Show that the only archimedean absolute value on $\mathbf{C}$ which extends $|\ |_{\mathbf{R}}$ is $|\ |_{\mathbf{C}}^{1/2}$. [Hint: show that there is a constant $C$ such that $|z| \leq C|z|_{\mathbf{C}}^{1/2}$ for all $z \in \mathbf{C}$.]

5) Let $v$ be a valuation on a field $K$ and let $\widehat{K}$ denote the completion of $K$ with respect to a corresponding absolute value. How does $v(\widehat{K})$ compare to $v(K)$? How does $k(\widehat{K}, v)$ compare to $k(K, v)$?

6) Suppose that a field $K$ is complete with respect to a non-archimedean absolute value $|\ |$. Suppose also that $\alpha_{i,j} \in K$ for $i, j \in \mathbf{Z}_{>0}$ satisfy

$$|\alpha_{i,j}| \longrightarrow 0$$

as $i, j \to \infty$. (That is given $\epsilon > 0$ there exists $I$ such that $|\alpha_{ij}| < \epsilon$ whenever $i$ or $j$ is $\geq I$.) Show that both the sums

$$\sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} \alpha_{i,j} \right)$$

and

$$\sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty}\alpha_{i,j}\right)$$

converge and that they converge to the same element of $K$.

7) Suppose that $K \supset \mathbf{R}$ and that $|\ |$ is an absolute value on $K$ extending $|\ |_{\mathbf{R}}$ which makes $K$ complete.

(a) Suppose also that $\alpha \in K$ satisfies

$$|\alpha^2 + 1| < 4/5.$$

Write $F(X) = X^2 + 1$. Define a sequence of elements $\alpha_n$ of $K$ by $\alpha_1 = \alpha$ and

$$\alpha_{n+1} = \alpha_n - F(\alpha_n)/F'(\alpha_n)$$

(where $F'(X) = 2X$ denotes the derivative of $F$). Show that

$$|\alpha_{n+1}^2 + 1| = |\alpha_{n+1} - \alpha_n|^2 \leq |\alpha_n^2 + 1|^2/4(1 - |1 + \alpha_n^2|)$$

and hence that, if $|\alpha^2 + 1| = 4\eta/5$, then

$$|\alpha_{n+1}^2 + 1| = |\alpha_{n+1} - \alpha_n|^2 \leq (4/5)\eta^{2^n}.$$

Deduce that $(\alpha_n)$ has a limit in $K$ and that this limit is a square root of $-1$. [This is Newton's method for approximating zeros.]

(b) Suppose now that $K$ does not contain $\mathbf{C}$. If $\alpha, \beta \in K$ and

$$|\alpha^2 + \beta^2| < 1$$

deduce from part (a) that $|\alpha|^2 < 5/4$ and $|\beta|^2 < 5/4$. Show that the function $|\ |'$ on $K(\sqrt{-1})$ defined by

$$|\alpha + \beta\sqrt{-1}|' = (\alpha^2 + \beta^2)^{1/2}$$

is an absolute value extending $|\ |$.

(c) Show that the only fields complete with respect to an Archimedean absolute value are $\mathbf{R}$ and $\mathbf{C}$ with an absolute value equivalent to $|\ |_{\mathbf{R}}$ and $|\ |_{\mathbf{C}}$. [Hint: First reduce to the case that the field contains a square root of $-1$. Show that there is an embedding $\mathbf{R} \hookrightarrow K$ compatible with the absolute values. Then show that there is an embedding $\mathbf{C} \hookrightarrow K$ compatible with the absolute values.]

8) Explain how to find a rational number $\alpha$ with $|\alpha^2 + 1|_5$ arbitrarily small. Use your algorithm to find $\alpha \in \mathbf{Q}$ with $|\alpha^2 + 1|_5 < 0.00001$. [Hint: $|2^2 + 1|_5 = 1/5 < 4/5$.]

9*) (a) Suppose that $K$ is an algebraically closed field of characteristic zero with the same cardinality as $\mathbf{C}$. Show that we can find a subset $\Omega \subset K$ with the same cardinality as $\mathbf{C}$ such that $K$ is algebraic over $\mathbf{Q}(\Omega)$ while the elements of $\Omega$ are algebraically independent over $\mathbf{Q}$. Deduce that $K \cong \mathbf{C}$ as fields.

(b) Show that $\mathbf{Q}_p$ has the same cardinality as $\mathbf{C}$. Show also that the algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$ has the same cardinality as $\mathbf{C}$. Deduce that $\overline{\mathbf{Q}}_p \cong \mathbf{C}$ as fields.

10*) Let $\mathcal{A}$ denote the set of formal sums

$$\sum_{n=-\infty}^{\infty} a_n T^n,$$

where $a_n \in \mathbf{Q}_p$ have $|a_n|_p$ bounded for $n \in \mathbf{Z}$ and

$$|a_n|_p \longrightarrow 0$$

as $n \to -\infty$. If

$$f(T) = \sum_{n=-\infty}^{\infty} a_n T^n \in \mathcal{A}$$

define

$$|f(T)| = \sup_n |a_n|_p.$$

Show that the 'obvious' definitions of addition and multiplication makes sense for elements of $\mathcal{A}$ (i.e.

$$\sum_n a_n T^n + \sum_n b_n T^n = \sum_n (a_n + b_n) T^n$$

and

$$(\sum_n a_n T^n)(\sum_n b_n T^n) = \sum_n (\sum_m a_m b_{n-m}) T^n)$$

and that this makes $\mathcal{A}$ into a field with absolute value $|\ |$. [Hint: To show that

$$0 \neq f = \sum_n a_n T^n \in \mathcal{A}$$

has a multiplicative inverse reduce to the case that $a_i \in \mathbf{Z}_p$ for all $i$, that $a_i \in p\mathbf{Z}_p$ for all $i < 0$ and $a_0 = 1$. Then consider the sub-case $a_i = 0$ for all $i < 0$. Use this sub-case to reduce further to the second sub-case $a_i \in p\mathbf{Z}_p$ for all $i \neq 0$.] Also show that $\mathcal{A}$ is complete with respect to $|\ |$. What is the residue field of $\mathcal{A}$?

11*) Let $v$ be a valuation on a field $K$. Show that the topology $| \ |_v$ induces on $K$ is locally compact if and only if

- the residue field $k(v)$ is finite,
- $v$ is discrete, and
- $K$ is complete with respect to $| \ |_v$.

Show further that in this case $\mathcal{O}_{K,v}$ is compact. [Hint: If the three itemized conditions are satisfied show that $\mathcal{O}_{K,v}$ is sequentially compact and deduce that $\mathcal{O}_{K,v}$ is compact and $K$ is locally compact.]

12*) Suppose that $p > 2$ is prime.

(a) If $A \in GL_n(\mathbf{Z}_p)$ satisfies $A \equiv I_n \bmod p$. If $A$ has finite order show that $A = I_n$. [Hint: Consider the case that $A$ has prime order $q$ make a binomial expansion of $(I_n + pB)^q$. Treat the cases $q = p$ and $q \neq p$ separately.]

(b) Show that $GL_n(\mathbf{F}_p)$ has order

$$(p^n - 1)(p^n - p)(p^n - p^2)...(p^n - p^{n-1}).$$

[Hint: Consider first the number of possibilities for the first column of an element of $GL_n(\mathbf{F}_p)$. Given the choice of first column, then consider the number of possibilities for the second column, and so on.]

(b) If $G$ is a finite subgroup of $GL_n(\mathbf{Z}_p)$ show that

$$\#G|(p^n - 1)(p^n - p)(p^n - p^2)...(p^n - p^{n-1}).$$

(c) If $G$ is a finite subgroup of $GL_n(\mathbf{Q})$ show that

$$\#G|\prod_q q^{b_q}$$

where, for $q$ an odd prime,

$$b_q = [n/(q-1)] + [n/q(q-1)] + [n/q^2(q-1)] + ...$$

and where

$$b_2 = n + 2[n/2] + [n/4] + [n/8] + ...$$

[Here we use $[\alpha]$ to denote the greatest integer less than or equal to a real number $\alpha$. You may use Dirichlet's theorem that there are infinitely many primes in an arithmetic progression whose common difference is coprime to any term.]

(d) Specialize this to conclude that if $G \subset GL_2(\mathbf{Q})$ is a finite subgroup then $\#G|24$.