Homework #2 for Math 136

Due date: Wednesday Oct 5, 2004

From Hsiung's book:
Page 96: #3(b), #8.
Page 108: #1(b).

Additional problems:

1. Consider the map:

$$\vec{X}(t) = \begin{cases} (t, 0, e^{-1/t^2}) & t > 0, \\ (0,0,0), & t = 0, \\ (t, 0, e^{-1/t^2}), & t < 0. \end{cases} \tag{1}$$

(a). Show that $\vec{X}(t)$ is differential everywhere.

(b). Show that $\vec{X}(t)$ is regular curve with curvature $\kappa(t) \neq 0$ for $t \neq 0, \sqrt{2/3}, -\sqrt{2/3}$ and $\kappa(0) = 0$.

(c). Show that the limit of the osculating plane as $t \to 0$, $t > 0$ is the plane $y = 0$. But the limit of the the osculating plane as $t \to 0$, $t < 0$ is the plane $z = 0$(recall that the osculating plane is the plane spanned by $\vec{e}_1$ and $\vec{e}_2$).

(d). Show that the torsion τ can be defined at $t = 0$. And the torsion function is identically zero(but obviously, the curve is NOT a plane curve).

2. Let $\vec{X} : I \to \mathbb{R}^3$ be a parametrized regular curve(not necessarily by arc length) with $\kappa \neq 0$ and $\tau(t) \neq 0$ for $t \in I$. The curve $\vec{X}$ is called a *Bertrand curve* if there exists a curve $\vec{Y} : I \to \mathbb{R}^3$ such that the principal normal lines of $\vec{X}$ and $\vec{Y}$ are equal. In this case, $\vec{Y}$ is called a *Bertrand mate* of $\vec{X}$, and we can write

$$\vec{Y}(t) = \vec{X}(t) + r(t)\vec{n}(t),$$

(Recall that the principal normal line is the line with the direction the of the principal normal vector of $\vec{e}_2$, in the standard notation of the Frenet frame. In the equation above, we use $\vec{n}$ to denote the principal normal vector)

Prove that

(a). $r(t)$ is a constant.

(b). $\vec{X}$ is a Bertrand curve if and only if there is a linear relation

$$A\kappa(t) + B\tau(t) = 1, \quad t \in I,$$

where A and B are nonzero constants and k and τ are the curvature and torsion of $\vec{X}$ respectively.

(c). If $\vec{X}$ has more than one Bertand mate, it has infinitely many Betrand mates. This case occurs if and only if both k and τ are constants.