Symplectic Geometry
Lecture 12

More on moment maps

Review: symplectic actions of a group.

Let (M, ω) be a symplectic manifold. A G-action $g \mapsto \mathcal{A}_g$ on M is called **symplectic** if $\mathcal{A}_g \in \mathrm{Symp}(M)$ for all $g \in G$. In other words, if

$$\mathcal{A}_g^* \omega = \omega \quad \forall\, g \in G.$$

Similarly, an action of a Lie algebra $\mathfrak{g}$ is called symplectic if $A_M \in \mathfrak{X}(M, \omega)$ for all $A \in \mathfrak{g}$. Clearly, if $\mathfrak{g}$ is the Lie algebra of G, then the $\mathfrak{g}$ action defined by a symplectic G action is symplectic.

Review: Weakly Hamiltonian actions

A symplectic G-action or a $\mathfrak{g}$-action is called **weakly Hamiltonian** if all the vector fields A_M are Hamiltonian. In other words, if for each $A \in \mathfrak{g}$ there is a smooth function $\Phi(A)$ on M such that

$$A_M = X_{\Phi(A)}.$$

One can always choose $\Phi(A)$ to depend linearly on A, by fixing the values of $\Phi(A_i)$ for the A_i in a basis of $\mathfrak{g}$ and then extending linearly. Then the map

$$A \mapsto \Phi(A)$$

can be viewed as a $\mathfrak{g}^*$ valued function on M:

$$\Phi \in C^\infty(M) \otimes \mathfrak{g}^*.$$

Review: Moment maps.

In other words, a symplectic G-action on a symplectic manifold (M, ω) is weakly Hamiltonian if there is a smooth map, called the **moment map**

$$\Phi : M \to \mathfrak{g}^*$$

i.e. $\Phi \in C^\infty(M) \otimes \mathfrak{g}^*$ such that

$$i(A_M)\omega = d\langle \Phi, A \rangle \quad \forall\ A \in \mathfrak{g}. \tag{4}$$

Review: Hamiltonian actions.

Definition 4 *A weakly Hamiltonian G-action is called* **Hamiltonian** *with moment map Φ if the moment map can be (and has been) chosen so as to be an equivariant map from M to $\mathfrak{g}^*$ relative to the co-adjoint action of G on $\mathfrak{g}^*$.*

Similarly one defines moment maps for Hamiltonian $\mathfrak{g}$ actions: one requires Φ to be $\mathfrak{g}$ equivariant in this case.

Review: Exact symplectic actions.

Suppose that (M, ω) is exact with $\omega = -d\alpha$ and that our group action preserves α. Then

$$D_{A_M}\alpha = 0 \quad \forall\ A \in \mathfrak{g}^*$$

and by Weil's formula

$$i(A_M)d\theta + di(A_M)\theta$$

which says that if we define $\Phi : M \to \mathfrak{g}^*$ by

$$\langle \Phi, A \rangle := \langle \alpha, A_M \rangle \tag{6}$$

then Φ is a moment map.

Review: Induced cotangent actions.

For example, if we are given a G-action on manifold Q we get a Hamiltonian G-action on the cotangent bundle T^*Q where now $\alpha = \alpha_Q$ is the fundamental one form of the cotangent bundle. The definition (6) now reads

$$\langle \Phi, A \rangle (q, \xi) = \langle \xi, A_Q(q) \rangle \quad \xi \in T_q^*(Q), \ A \in \mathfrak{g}. \tag{7}$$

$\mathbb{C}^n$

Consider $\mathbb{C}^n$ with its standard complex coordinates $z_1, \ldots, z_n$ and write $z_j = q_j + ip_j$ and

$$\omega := \sum_j dq_j \wedge dp_j = \frac{i}{2} \sum_j dz_j \wedge d\overline{z}_j = \frac{i}{2} h(dz, dz)$$

where

$$h(u, w) = \sum_j u_j \overline{w}_j$$

is the Hermitian form. Let α be the one form on $\mathbb{C}^n$ defined by

$$\alpha := \frac{i}{4} \sum_j \left(\overline{z}_j dz_j - z_j d\overline{z}_j\right).$$

Then

$$d\alpha = -\omega.$$

The unitary group.

$$\alpha := \frac{i}{4}\sum_j (\overline{z}_j dz_j - z_j d\overline{z}_j)\,.$$

Then

$$d\alpha = -\omega.$$

We can write

$$\alpha = \frac{1}{2}\operatorname{Im} h(z, dz).$$

In this form it is clear that α is preserved by the unitary group $U(n)$ and hence the action of $U(n)$ on $\mathbb{C}^n$ is Hamiltonian with moment map given by

$$\langle \Phi, A \rangle = \langle \alpha, A_{\mathbb{C}^n} \rangle,$$

$$\langle \Phi, A \rangle = \langle \alpha, A_{\mathbb{C}^n} \rangle,$$

where $A \in \mathfrak{u}(n)$ is a skew adjoint matrix. So we must compute $A_{\mathbb{C}^n}$. Now

$$\exp -tA : \; z \mapsto z - tAz + \cdots$$

so

$$z_k(t) = z_k - t \sum_\ell A_{k\ell}(q_\ell + ip_\ell) + \cdots$$

so the vector field $A_{\mathbb{C}^n}$ is given by

$$\sum_k \left(\sum_\ell (-\operatorname{Re} A_{k\ell} q_\ell + \operatorname{Im} A_{k\ell} p_\ell) \right) \frac{\partial}{\partial q_k} +$$

$$\sum_k \left(\sum_\ell (-\operatorname{Re} A_{k\ell} p_\ell - \operatorname{Im} A_{k\ell} q_\ell) \right) \frac{\partial}{\partial p_k}.$$

We can give this a more convenient form by introducing the complex vector fields

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial q_j} - i\frac{\partial}{\partial p_j}\right), \quad \frac{\partial}{\partial \overline{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial q_j} + i\frac{\partial}{\partial p_j}\right).$$

Notice that if

$$c = a + ib, \quad w = x + iy$$

then

$$cw\frac{\partial}{\partial z} + \overline{cw}\frac{\partial}{\partial \overline{z}} =$$

$$\frac{1}{2}\left[(a+ib)(x+iy)\left(\frac{\partial}{\partial q} - i\frac{\partial}{\partial p}\right) + (a-ib)(x-iy)\left(\frac{\partial}{\partial q} + i\frac{\partial}{\partial p}\right)\right] =$$

$$= (ax - by)\frac{\partial}{\partial q} + (ay + bx)\frac{\partial}{\partial p}.$$

If $c = a + ib, \quad w = x + iy$ then $$cw\frac{\partial}{\partial z} + \overline{cw}\frac{\partial}{\partial \overline{z}} =$$

$$(ax - by)\frac{\partial}{\partial q} + (ay + bx)\frac{\partial}{\partial p}.$$

$$\sum_k \left(\sum_\ell (-\operatorname{Re} A_{k\ell} q_\ell + \operatorname{Im} A_{k\ell} p_\ell) \right) \frac{\partial}{\partial q_k} +$$

$$\sum_k \left(\sum_\ell (-\operatorname{Re} A_{k\ell} p_\ell - \operatorname{Im} A_{k\ell} q_\ell) \right) \frac{\partial}{\partial p_k}.$$

write the above expression for $A_{\mathbb{C}^n}$ as

$$-\sum_{k\ell} A_{k\ell} z_\ell \frac{\partial}{\partial z_k} - \sum_{k\ell} \overline{A}_{k\ell} \overline{z}_\ell \frac{\partial}{\partial \overline{z}_k}.$$

The fact that A is skew Hermitian allows us to rewrite this as

$$-\sum_{k\ell} A_{k\ell} z_\ell \frac{\partial}{\partial z_k} + \sum_{k\ell} A_{\ell k} \overline{z}_\ell \frac{\partial}{\partial \overline{z}_k}.$$

$$A_{\mathbb{C}^n} \quad = \quad -\sum_{k\ell} A_{k\ell} z_\ell \frac{\partial}{\partial z_k} + \sum_{k\ell} A_{\ell k} \overline{z}_\ell \frac{\partial}{\partial \overline{z}_k}.$$

Taking interior product with

$$\frac{i}{4} \sum_j (\overline{z}_j dz_j - z_j d\overline{z}_j)$$

gives

$$-\frac{i}{2} \sum_{k\ell} A_{k\ell} z_\ell \overline{z}_k = -\frac{i}{2} h(Az, z).$$

Of course, by restriction, this applies to any subalgebra of $\mathfrak{u}(n)$. So we have proved:

The moment map for *U(n)* acting on $\mathbf{C}^n$.

Theorem 1 *Let G be any subgroup of $U(n)$ with Lie algebra $\mathfrak{g}$. Then the action of G on $\mathbb{C}^n$ is Hamiltonian with moment map*

$$\Phi : \mathbb{C}^n \to \mathfrak{g}^*$$

given by

$$\langle \Phi(z), A \rangle = -\frac{i}{2} \sum_{k\ell} A_{k\ell} z_\ell \overline{z}_k, \quad z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{C}^n, \quad A \in \mathfrak{g}. \quad (1)$$

The maximal torus of *U(n)* .

Consider the subgroup of $U(n)$ consisting of all diagonal matrices. As a group it is $\mathbb{T}^n$, the n-dimensional torus. The Lie algebra of this subgroup is the space of all diagonal matrices with purely imaginary entries. So, as a vector space, this is $\mathbb{R}^n$ with the element

$$\mathbf{r} = \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}$$

identified with the diagonal matrix

$$\mathrm{diag}(ir_1, \ldots, ir_n),$$

The image of the moment map.

The theorem says that the moment map $\Phi : \mathbb{C}^n \to \mathbb{R}^n$ is given by

$$\Phi(\mathbf{z}) = \frac{1}{2}(|z_1|^2, \ldots, |z_n|^2) \tag{2}$$

if

$$\mathbf{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}.$$

Notice that the image of the moment map is the entire first orthant of $\mathbb{R}^n$ consisting of those vectors whose entries are non-negative.

Complex representations of a torus.

Let $\mathbf{T} = \mathbb{T}^d$ be a d-dimensional torus, and suppose that we are given a representation of $\mathbf{T}$ on an n-dimensional complex vector space V. From group theory we know that V is isomorphic to a direct sum of one dimensional complex representation spaces. The one dimensional representations of $\mathbf{T}$ can be described as follows:

Weights.

The Lie algebra $\mathfrak{t}$ of $\mathbf{T}$ is a d-dimensional real vector space. The exponential map

$$\exp : \ \mathfrak{t} \to \mathbf{T}$$

has as its kernel a d-dimensional lattice Λ. If $\beta \in \mathfrak{t}^*$, then the

$$\exp x \mapsto e^{i\langle \beta, x\rangle}$$

is well defined if and only if $\langle \beta, \lambda \rangle \in 2\pi\mathbb{Z}$for all $\lambda \in \Lambda$. This condition picks out a lattice in $\mathfrak{t}^*$ called the **weight lattice**. The elements of the weight lattice are called **weights**. Then every one dimensional representation of $\mathbf{T}$ is of the above form with β a weight.

So if we are given a representation of $\mathbf{T}$ on a complex vector space V, then we can choose a basis of V which identifies V with $\mathbb{C}^n$ and such that all the elements of T are represented by diagonal unitary matrices.

The moment map.

This means that we are given a homomorphism of $T \to U(n)$ whose induced homomorphism on the Lie algebra level sends

$$\mathfrak{t} \ni x \mapsto i\mathrm{diag}(\beta_1(x), \ldots, \beta_n(x)).$$

The transpose of this map sends the row vector $(u_1, \ldots, u_n)$ to

$$u_1\beta_1 + \cdots + u_n\beta_n.$$

So the moment map $V \to \mathfrak{t}^*$ in terms of the identification of V with $\mathbb{C}^n$ is given by

$$\Phi(\mathbf{z}) = \frac{1}{2}|z|_1^2\beta_1 + \cdots + \frac{1}{2}|z_n|^2\beta_n. \tag{3}$$

Thus the image of the moment map is the convex cone consisting of all non-negative linear combinations of the β_i.

Preview.

$$\Phi(\mathbf{z}) = \frac{1}{2}|z|_1^2\beta_1 + \cdots + \frac{1}{2}|z_n|^2\beta_n. \tag{3}$$

Thus the image of the moment map is the convex cone consisting of all non-negative linear combinations of the β_i. We will denote this cone by $C(\beta_1, \ldots, \beta_n)$.

We will make use of this fact to prove that the image of a the moment map for a Hamiltonian action of a torus on a compact manifold is a convex polytope.

Normalization near a fixed point.

We can make some preliminary steps in this direction right now. Let $\mathbf{T}$ act in Hamiltonian fashion on a manifold M and let x be a fixed point of $\mathbf{T}$. Then we get a linear representation of $\mathbf{T}$ on the tangent space T_xM. We can put a $\mathbf{T}$-invariant Riemann metric on M and then a compatible complex structure on T_xM. We can now apply Darboux's theorem, making sure that all choices are $\mathbf{T}$-equivariant. Thus there will be a $\mathbf{T}$-invariant neighborhood O of the origin in T_xM and a symplectomorphism of O onto a a $\mathbf{T}$-invariant neighborhood of x in M intertwining the two $\mathbf{T}$ actions. The moment maps for these two actions must be intertwined up to an additive constant. Since the constant term in (3) is zero, the additive constant is just $\Phi_M(x)$.

The cone near a fixed point.

From the explicit nature of the map in (3) we see that the image of any open neighborhood O of the origin under (3) is the intersection of an open set about 0 in $\mathfrak{t}^*$ with $C(\beta_1, \ldots, \beta_n)$. So we have proved:

Proposition 1 *Let (M, ω, Φ_M) be a Hamiltonian $\mathbf{T}$-space where $\mathbf{T}$ is a torus, and let x be a fixed point of $\mathbf{T}$. Let $\beta_1, \ldots, \beta_n$ be the weights for the representation of $\mathbf{T}$ on T_xM relative to a compatible complex structure on T_xM. Then there is a $\mathbf{T}$ invariant neighborhood U of x in M and a neighborhood U' of $\Phi_M(x)$ in $\mathfrak{t}^*$ such that*

$$\Phi_M(U) = U' \cap (\Phi_M(x) + C(\beta_1, \ldots, \beta_n)).$$

The local cone at a general point.

Now let x be any point of M, not necessarily a fixed point. It will have an isotropy subgroup $\mathbf{T}_x \subset \mathbf{T}$ and the connected component $\mathbf{T}_x^0$ of the identity in $\mathbf{T}_x$ will be a torus. The Lie algebra $\mathfrak{t}_x$ of this torus is a subalgebra of the Lie algebra $\mathfrak{t}$ of $\mathbf{T}$. We let

$\iota_x : \mathfrak{t}_x \to \mathfrak{t}$ denote the injection and $\pi_x : \mathfrak{t}^* \to \mathfrak{t}_x^*$

the dual projection. Then

$$\pi_x \circ \Phi_M$$

is a moment map for the action of $\mathbf{T}_x^0$ on M and x is a fixed point for this action.

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is a moment map for the action of $\mathbf{T}_x^0$ on M and x is a fixed point for this action. So we may apply Prop. 1 to conclude that there are neighborhoods U of x and U' of $\pi_x \circ \Phi_M$ in $\mathfrak{t}_x^*$ such that

$$\pi \circ \Phi_M(U) = U' \cap (\Phi_M(x) + C(\beta_1, \dots, \beta_n)).$$

Let

$$O' := \pi_x^{-1}(U')$$

and

$$C'_x(\beta_1, \dots, \beta_n) = \pi_x^{-1}(C(\beta_1, \dots, \beta_n)).$$

So O' is an open set in $\mathfrak{t}^*$ and $C'_x(\beta_1, \dots, \beta_n)$ is a cone in $\mathfrak{t}^*$. (Of course this cone will contain the subspace $\ker \pi_x$.) Let $O := \Phi_M^{-1}(O')$. We will prove, using a normal form for the moment map, that

Theorem 2 *Let (M, ω, Φ_M) be a Hamiltonian $\mathbf{T}$ space where $\mathbf{T}$ is a torus. Then every $x \in M$ has a neighborhood O such that there is an open set O' in $\mathfrak{t}^*$ and a cone $C(x)$ in $\mathfrak{t}^*$ such that*

$$\Phi(O) = O' \cap C(x).$$

More precisely, the cone $C(x)$ is

$$C(x) := \Phi_m(x) + \left\{ \mu \in \mathfrak{t}^* \,\middle|\, \mu_{|\mathfrak{t}_x} = \sum r_i \beta_i, \ r_i \geq 0 \right\}$$

where the β_i are the weights of the representation of $\mathbf{T}_x^0$ on $T_x M$.

The proof of the normal form theorem will use the co-isotropic embedding theorem.

I will come back to this theorem in Lecture 14 where we will use this theorem to prove that if M is compact, the image of its moment map is a convex polyhedron.

A circle action on $\mathbf{C}^n$.

Consider the extreme special case where $d = 1$ and $\beta = \beta_1$ is simply

$$\langle \beta, x \rangle = x.$$

This corresponds to the diagonal subgroup of $U(n)$ with the same entry e^{it} at all positions along the diagonal, in other words, simultaneous rotation in each of one dimensional complex subspaces (which are planes over the real numbers). So under the identification of $\mathfrak{t}^*$ with $\mathbb{R}$, the moment map is given by

$$\Phi(\mathbf{z}) = \frac{1}{2}\|\mathbf{z}\|^2.$$

The spheres.

Suppose we fix non-zero value μ of the moment map, say $\mu = \frac{1}{2}$. Then $\Phi^{-1}(\mu)$ is the sphere Q consisting of all $\mathbf{z}$ with

$$\|\mathbf{z}\|^2 = 1.$$

The vector field $X := A_{\mathbb{C}^n}$ is given by

$$X = i \sum_j \left(-z_j \frac{\partial}{\partial z_j} + \overline{z}_j \frac{\partial}{\partial \overline{z}_j} \right).$$

Now $D_X \alpha = 0$ and $\Phi = i(X)\alpha$ is a constant when restricted to Q. So $i(X)\omega = 0$ when restricted to Q. In other words, at each $\mathbf{z} \in Q$,

$$T_{\mathbf{z}} Q \subset X_{\mathbf{z}}^{\perp}.$$

The spheres are co-isotropic

$$T_{\mathbf{z}}Q \subset X_{\mathbf{z}}^{\perp}.$$

Since the real dimension of Q is $2n-1$, we see that

$$T_{\mathbf{z}}Q = X_{\mathbf{z}}^{\perp}$$

and hence

$$T_{\mathbf{z}}Q^{\perp} = \mathbb{R} \cdot X_{\mathbf{z}},$$

the line in $T_{\mathbf{z}}Q$ spanned by $X_{\mathbf{z}}$ and this line is contained in $T_{\mathbf{z}}Q$. So we see that Q is a co-isotropic submanifold whose null-foliation is spanned at each point by X.

The foliation is a fibration.

So we see that Q is a co-isotropic submanifold whose null-foliation is spanned at each point by X. The leaves of this null foliation are just the circles consisting of the orbits of our subgroup, i.e. points of the form $e^{i\theta}\mathbf{z}$ as θ varies and $\mathbf{z} \in Q$ is fixed. This foliation is fibrating, and we can identify the base as complex $n-1$ dimensional projective space. Indeed, $\mathbb{CP}^{n-1}$ is defined as the space of all (complex) lines in $\mathbb{C}^n$ passing through the origin. Each such line intersects Q precisely in a circle of the type just described, and hence we see that our foliation in fibrating with base $B = \mathbb{CP}^{n-1}$.

The symplectic structure on projective space.

Each such line intersects Q precisely in a circle of the type just described, and hence we see that our foliation in fibrating with base $B = \mathbb{CP}^{n-1}$. We conclude that $\mathbb{CP}^{n-1}$ carries a symplectic structure which is invariant under the action of $U(n)$. Since $U(n)$ acts transitively on $\mathbb{CP}^{n-1}$ and the isotropy subgroup which is conjugate to $U(n-1)$ acts irreducibly on the tangent space at each point , we conclude that any two invariant symplectic forms can only differ by an overall constant factor.

We will now embark on a generalization of this construction to general Hamiltonian group actions.

The derivative of the moment map.

Let us be given a Hamiltonian action of a Lie group G on a symplectic manifold (M, ω) with moment map $\Phi : M \to \mathfrak{g}^*$. The defining property of the moment map is

$$d\langle \Phi, A \rangle = i(A_M)\omega \quad \forall\ A \in \mathfrak{g}.$$

In this equation, A is a constant (as a function on M) so we can rewrite this as

$$\langle d\Phi_m(v), A \rangle = \omega_m(A_M(m), v) \quad v \in T_m M, \ \ A \in \mathfrak{g}. \tag{4}$$

The evaluation map and its transpose.

$$\langle d\Phi_m(v), A\rangle = \omega_m(A_M(m), v) \quad v \in T_m M, \quad A \in \mathfrak{g}. \qquad (4)$$

Here is a way of thinking about this central equation: The action of G on M gives a linear map

$$\mathbf{ev}_M(m) : \mathfrak{g} \to T_m M, \qquad A \mapsto A_M(m)$$

for each $m \in M$. Let us call this map the **evaluation** map. The transpose of the evaluation map would be a linear map from $T_m^* M \to \mathfrak{g}^*$. The symplectic form ω_m gives us an isomorphism

$$T_m M \to T_m^* M, \quad v \mapsto \omega_m(\cdot, v).$$

The moment map and the evaluation map.

$$\langle d\Phi_m(v), A\rangle = \omega_m(A_M(m), v) \quad v \in T_mM, \quad A \in \mathfrak{g}. \qquad (4)$$

$$T_mM \to T_m^*M, \quad v \mapsto \omega_m(\cdot, v).$$

Using this isomorphism, we can regard the transpose as a map $T_mM \to \mathfrak{g}^*$ and (4) says that

Proposition 1 $d\Phi_m : T_mM \to \mathfrak{g}^*$ *is the transpose of the evaluation map* $\mathfrak{g} \to T_mM$ *when we identify* T_m^*M *with* T_mM *using* ω_m.

The kernel of the derivative of the moment map.

Let $\mathfrak{g}_M(m)$ denote the subspace of $T_m M$ consisting of all the $A_M(m)$, $A \in \mathfrak{g}$. So $\mathfrak{g}_M(m)$ is the image of the evaluation map at m. Geometrically, it is the tangent space to the orbit $G \cdot m$ at m. Since the kernel of the transpose of a linear map is the annihilator space of the image, we conclude that

$$\ker d\Phi_m = \mathfrak{g}_M(m)^{\perp} \tag{5}$$

where $\perp$ means the perpendicular relative to ω_m.

A transitive Hamiltonian space covers a coadjoint orbit.

For example, suppose that G acts transitively on M so that $\mathfrak{g}_M(m) = T_m M$ at all points. Then the kernel of $d\Phi_m$ is $\{0\}$ at all m. In other words, Φ is an immersion. Since Φ is equivariant and G acts transitively on M, the image of Φ must be a single orbit $\mathcal{O}$ of the co-adjoint action of G on $\mathfrak{g}^*$:

$$\Phi(M) = \mathcal{O}, \quad \mathcal{O} = G \cdot \Phi(m), \quad m \in M.$$

So the map

$$\Phi : M \to \mathcal{O}$$

is a covering map.

The restriction of the symplectic form to an orbit.

Now observe that for *any* Hamiltonian G-action, the moment map determines the restriction of the symplectic form ω to any orbit of G acting on M. Indeed, The tangent space to the orbit at any point m consists of tangent vectors of the form $A_M(m)$ and

$$\omega(A_M, B_M) = -\{\Phi^A, \Phi^B\} = -\Phi^{[A,B]}$$

where we recall the notation $\Phi^A := \langle \Phi, A \rangle$.

The Kostant-Souiau form, I.

Applied to the case of a transitive Hamiltonian action, this says that the moment map completely determines the symplectic form in the transitive case. Since $\Phi : M \to \mathcal{O}$ is a covering map, this suggests that there is a unique symplectic form on $\mathcal{O}$ such that the action of G on $\mathcal{O}$ is Hamiltonian with moment map consisting of he injection ι of $\mathcal{O}$ as a submanifold of $\mathfrak{g}^*$. From the preceding discussion we know that if such a form exists, it is unique.

We now write down the symplectic form which is known as the **Kostant-Souriau** form on $\mathcal{O}$:

The Kostant-Souiau form, 2.

Each $\mu \in \mathfrak{g}^*$ determines a skew-symmetric form $\mathbf{B}_\mu$ on $\mathfrak{g}$ defined by

$$\mathbf{B}_\mu(A, B) := \langle \mu, [A, B] \rangle.$$

We can write this as

$$\mathbf{B}_\mu(A, B) = \langle \mu, \mathrm{ad}(A)B \rangle = \langle (\mathrm{ad}\, A)^* \mu, B \rangle = \langle A_{\mathcal{O}}(\mu), B \rangle$$

where $\mathcal{O}$ is the G-orbit through μ and $A_{\mathcal{O}}$ is the generating vector field on $\mathcal{O}$ corresponding to $A \in \mathfrak{g}$. So we see that the kernel of $\mathbf{B}_\mu$ consists precisely of those A such that $A_{\mathcal{O}}(\mu) = 0$. In other words, the anti-symmetric two form σ_μ on $T_\mu\mathcal{O}$ determined by

$$\sigma_\mu(A_{\mathcal{O}}(\mu), B_{\mathcal{O}}(m)) := \mathbf{B}_\mu(A, B)$$

is well defined.

The Kostant-Souiau form, 3.

$$\mathbf{B}_\mu(A,B) = \langle \mu, \mathrm{ad}(A)B\rangle = \langle (\mathrm{ad}\, A)^*\mu, B\rangle = \langle A_{\mathcal{O}}(\mu), B\rangle$$

$$\sigma_\mu(A_{\mathcal{O}}(\mu), B_{\mathcal{O}}(m)) := \mathbf{B}_\mu(A,B)$$

is well defined. So we get a two form σ on $\mathcal{O}$. To check that this form is invariant under the action of G we must check that

$$\mathbf{B}_\mu(A,B) = \mathbf{B}_{g\cdot\mu}(\mathrm{Ad}_g\, A, \mathrm{Ad}_g\, B) \quad \forall\ A, B \in \mathfrak{g},\ \ g \in G.$$

But

$$\mathbf{B}_{g\cdot\mu}(\mathrm{Ad}_g\, A, \mathrm{Ad}_g\, B) = \langle g\cdot\mu, [\mathrm{Ad}_g\, A, \mathrm{Ad}_g\, B]\rangle$$

$$= \langle g\cdot\mu, \mathrm{Ad}_g\,([A,B])\rangle = \langle \mu, [A,B]\rangle = \mathbf{B}_\mu(A,B).$$

So σ is invariant (and hence smooth).

The Kostant-Souiau form, 4.

So we know that $D_{A_{\mathcal{O}}}\sigma = 0$. Let ι denote the inclusion map $\iota : \mathcal{O} \to \mathfrak{g}^*$. Then for any $A, B \in \mathfrak{g}$,

$$i(B_{\mathcal{O}})d\langle \iota, A\rangle = D_{B_{\mathcal{O}}} d\langle \iota, A\rangle = \langle (\mathrm{ad}^*_B) \circ \iota, A\rangle$$

$$= -\langle \iota, [A, B]\rangle = -\sigma(A_{\mathcal{O}}, B_{\mathcal{O}})$$

which would be the condition for ι to be the moment map if we knew that σ were closed. But the preceding equation implies that $i(A_{\mathcal{O}})\sigma = d\langle \iota, A\rangle$ is closed and hence by Weil's formula

$$i(A_{\mathcal{O}})d\sigma = D_{A_{\mathcal{O}}}\omega - d(i(A_{\mathcal{O}}\sigma) = 0$$

for all A. So $d\sigma = 0$ since the $A_{\mathcal{O}}$ span he tangent space to the orbit.

The Kostant-Souriau theorem.

Putting it all together we obtain:

Theorem 3 [Kostant-Souriau]. *Any co-adjoint orbit $\mathcal{O}$ carries a unique symplectic form σ for which the injection*

$$\iota : \mathcal{O} \to \mathfrak{g}^*$$

is the moment map. At each $\mu \in \mathcal{O}$ this symplectic form is given by

$$\sigma_\mu(A_{\mathcal{O}}(\mu), B_{\mathcal{O}}(\mu)) = \langle \mu, [A, B] \rangle.$$

If M is any symplectic manifold on which G acts in a Hamiltonian fashion and the action is transitive, then the moment map $\Phi : M \to \mathfrak{g}^*$ is in fact a covering map of some orbit $\mathcal{O}$ of G acting on $\mathfrak{g}^*$ and the symplectic form on G is the pull-back via Φ of the symplectic form σ on $\mathcal{O}$.

The image of the derivative of the moment map.

$d\Phi_m$ maps $T_m M$ to $T_{\Phi(m)}(\mathfrak{g}^*)$. Since $\mathfrak{g}^*$ is a vector space, we may identify $T_{\Phi(m)}(\mathfrak{g}^*)$ with $\mathfrak{g}^*$ and hence think of $d\Phi_m$ as a map

$$d\Phi_m : T_m M \to \mathfrak{g}^*.$$

The image of this map will be a subspace $\mathrm{im}(d\Phi_m) \subset \mathfrak{g}^*$. So the annihilator space $(\mathrm{im}\, d\Phi_m)^0$ of this subspace will be that subspace of $\mathfrak{g}$ consisting of all $A \in \mathfrak{g}$ such that $\langle \mu, A \rangle = 0$ for all $\mu \in \mathrm{im}\, d\Phi_m$. Prop.2 tells us that

$$(\mathrm{im}\, d\Phi_m)^0 = \{A \in \mathfrak{g} | A_M(m) = 0\}. \tag{6}$$

The stabilizer subgroup.

$$(\mathrm{im}\, d\Phi_m)^0 = \{A \in \mathfrak{g} | A_M(m) = 0\}. \tag{6}$$

The right hand side of this equation is the Lie algebra of the subgroup $G_m \subset G$ consisting of those elements which fix m sometimes called the **stabilizer group** of m. As a corollary of this equation we see that

Proposition 2 *$d\Phi_m$ is surjective if and only if the stabilizer subgroup of m is discrete.*