Odds and ends

Math 139

References: "Knots and Links" by Peter Cromwell, "The Knot Book" by Colin Adams, and "A Survey of Knot Theory" by Akio Kawauchi.

Brunnian Links are non-trivial.

In order to prove this we must detour and define tangle sum and non-split tangles. In these notes I'll denote a tangle by (B, t) where t is the tangle in the three ball B.

Definition. A *tangle sum* of (A, s) and (B, t) is the link $(A, s) \cup_\phi (B, t)$ obtained by gluing them together via a homeomorphism $\phi : \partial(B, t) \to \partial(A, s)$.

Exercise and Remark: 1) Find an example that shows the link type of a tangle sum is not uniquely determined by the tangles. 2) Construct a pair of distinct knots from the same pair of tangles by taking distinct tangle sums.

Definition. A tangle (B, t) is *non-split* if any proper disk D in B does not split t in B.

Remark: A trivial tangle is split, but a split tangle is not necessarily trivial. A split 2-tangle made of unknotted arcs is trivial. A non-split 2-tangle made of unknotted arcs is nontrivial.

Theorem 1. *Any link obtained by any tangle sum of two non-split tangles is non-split.*

Proof. Let T denote the common sphere of the tangle sum and let S denote the splitting sphere. Suppose that $T \cap S = \emptyset$, then $S \subset T$ or $S \subset (S^3 \setminus T)$. This means one tangle is split, a contradiction. Thus $T \cap S \neq \emptyset$ and is a set of nested loops, denote one by λ. Now λ bounds a disk on S and by construction $S \cap t = \emptyset$. Suppose λ bounds a disk Δ on T. If $t \cap \Delta = \emptyset$, then use surgery to simplify these intersections. We are left with a sphere T such that $t \cap \Delta \neq \emptyset$. There must be two points of intersection (as S is a splitting sphere). This means there is a strand of t in $T \cap S$, hence Δ is a splitting disk for (T, t), a contradiction. □

Exercise and Remark: This theorem and the previous remarks should give you the tools you need to prove the Brunnian links are non-trivial. Many of you have worked out how to construct Brunnian links and how each component relates to the others. Start with a 2-tangle that you can show is non-split. Made a tangle sum of this 2-tangle with itself to give a non-split 2 component link. Use induction. Further hints available on request.

Rational knots are prime.

You now have all the pieces needed to prove this result. I'll indicate some of the steps of the proof below.

Proof. Let K be the rational knot and R be the common sphere of the rational knot (so on each side of R there are trivial 2-tangles). Assume that the rational knot is not prime and let S be the factorizing sphere: $K \cap S$ in two points.

(1) Assume that $R \cap S = \emptyset$ and derive a contradiction.

(2) Hence $R \cap S \neq \emptyset$, it is a set of nested loops. The first step is to remove the simplest intersections by surgery. Let λ be an innermost loop on S which bounds a disk Δ and let λ bound a disk Δ' on R. Assume $K \cap \Delta = \emptyset$ and $\Delta' \cap K = \emptyset$...

(3) The rest of the proof is a matter of considering the intersection of K with Δ and Δ' and deriving contradictions. The contradictions might involve the non-triviality of K or its factors, or the triviality of the 2-tangles. For example, what happens when $K \cap \Delta = \emptyset$ but $\Delta' \cap K$ is 1 point or is 2 points? Now, keep going. You might find that there are sub-cases within these cases. Further hints available on request.

□