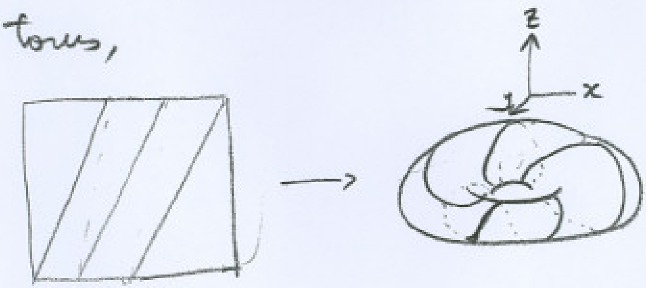
Problem Set 1

Problem 1

a) $T(n,1)$ are trivial because, thinking of it as the image of lines with slope n on a square under the quotienting map onto the torus,



we see that a projection onto the xy plane produces a diagram with no crossings.

Alternatively, you could check that the projection of the curve

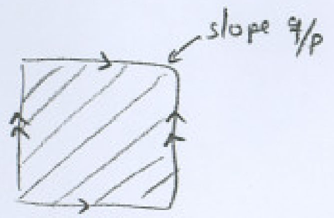
$$x = \left(2 + \cos\frac{\theta}{n}\right)\cos\theta$$

$$y = \left(2 \cos\frac{\theta}{n}\right)\sin\theta$$

$$z = \sin\frac{\theta}{n}$$

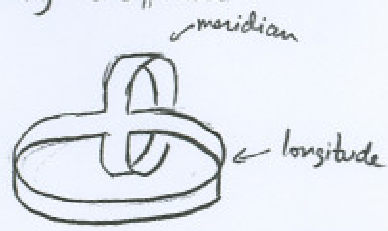
onto the xy plane produces the projection with no crossings.

b) A line with slope q/p determines the segments on a square which, under the projection map onto a torus, form the (p,q) torus.



Identifying in a different order, (double arrow, then single arrow) we reverse the roles of the meridian and parallel to get the (q,p) torus knot. The image of the line under both identifications must be the same, so the (p,q) torus knot and (q,p) torus knot are isotopic

Alternatively, you could remove a disk from the torus disjoint from the (p,q) torus knot, isotope until you get two annuli attached by a square



we can undo this transformation to exchange the meridian & longitude, getting the equivalence.

c) The claims about sign follow immediately from the parametrization of the torus knot

$$x = \left(2+\cos\left(\tfrac{q\theta}{p}\right)\right)\cos\theta$$
$$y = \left(2+\cos\left(\tfrac{q\theta}{p}\right)\right)\sin\theta$$
$$z = \sin\left(\tfrac{q\theta}{p}\right).$$

Problem 2

If $p(K_1) = 4$ then since

$$cr(K_1) \le \frac{1}{2}(p(K_1)-1)(p(K_1)-4)$$

$$\le \frac{3}{2}\cdot 0 = 0$$

so K_1 is the trivial knot.

If $p(K_1) = 5$, then

$cr(K_2) \le 2$ using the same formula.

But the three options $cr(K_2) = \begin{cases} 0 \\ 1 \\ 2 \end{cases}$ all yield trivial knots,

so K_2 must be trivial

Alternatively, for the first part, you could note that if the 4 points are not coplanar, then they form a tetrahedron in $\mathbb{R}^3$ and the ~~for~~ knot is a sequence of its edges. Since the tetrahedron is homeomorphic to S^2, this gives an embedding of the knot in $S^2 = \mathbb{R}^2 \cup \infty$, so the knot must be trivial.

Problem 3: Class exercise

Problem 4

We can construct a representative of the knot class by creating a height function: Pick a point x_0 and a direction on the projection that gives us the ascending property. If we have n crossings $c_1, \dots, c_n$ that we go through as we travel starting from x_0, let $t : [0, n+1] \to \mathbb{R}^3$ be a parametrization st $t(0) = t(n+1) = x_0$ and $t(c_i)$ has height i and projects onto c_i. Projecting onto the yz plane, we get a coil:

which represents the trivial knot by performing R1 + 22 moves from top to bottom to untwist the knot.

That every diagram can be changed to an ascending one is trivial, we obtain it by choosing a direction and switching crossings to overcrossings in whatever order we encounter them.

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Problem 5

Pick a diagram of K with $c(K)$ crossings. There is an orientation in which the number of crossings which we encounter first as overpasses is $\leq \frac{1}{2}c(K)$. If we switch all these crossings into underpasses, you obtain an ascending diagram. It follows that the unknotting # is $\leq \frac{1}{2}c(K)$

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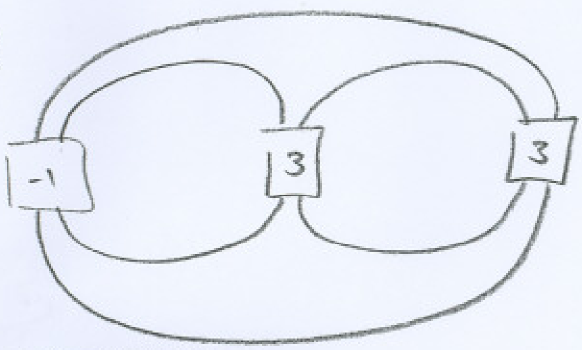
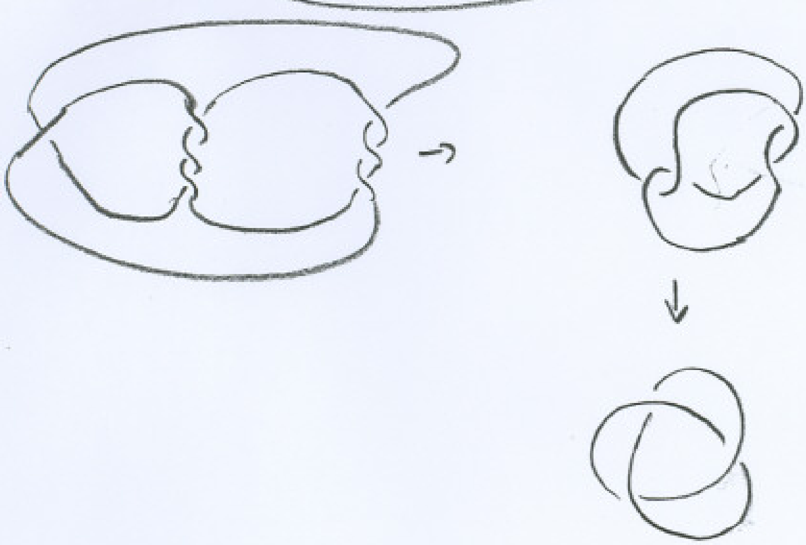
Problem 6





right-handed trefoil!

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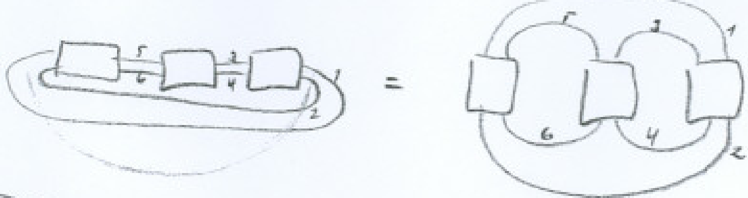
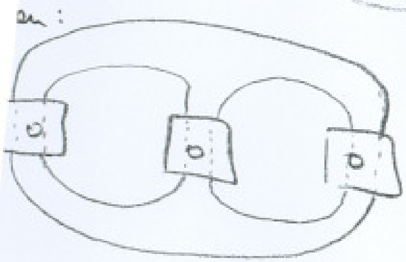
...suffices to try values of p, q, r mod 2, namely 0 or 1.

... we can try all possible options, noting from the diagram

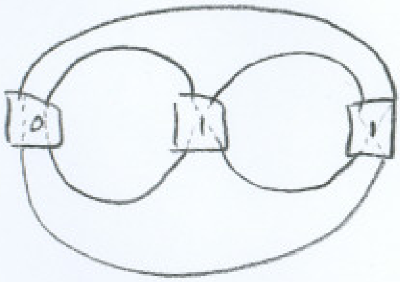
... that the ordering of mod 2 values is irrelevant:



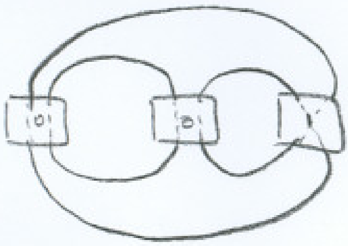
...en:



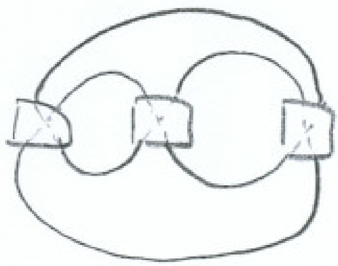
3 components.



1 component



2 comp.



1 component.