MATHEMATICS 152, FALL 2003
METHODS OF DISCRETE MATHEMATICS
Outline #3 (Groups)

This class introduces us to the formal mathematical structure of a group and is based around the first several sections of Biggs' Chapter 20.

1. Define a *group* as a set with a binary operation on that set satisfying four axioms. Give examples based on familiar number systems such as $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$, and explain which of the usual operations make these systems into groups. (In particular, note that all of these are groups under addition, while only the latter two are groups under multiplication and then only if 0 is deleted from the set.) (Section 20.1)

2. Invent examples of sets with binary operations that are not groups, illustrating which particular axioms fail. Try to find one example for each axiom.

3. Show that the collection of permutations on a fixed set of n elements, denoted S_n, is a group when the operation in question is the composition of functions. Define the *cardinality* of a group G to be the number of elements in the group, denoted $|G|$, and show that $|S_n| = n!$. (Sections 10.6 and 21.1.)

4. Classify the symmetries of the equilateral triangle by type, and construct a table of compositions of these symmetries, showing that in fact they form a group under composition, denoted D_3. Define an *abelian* group, and show that D_3 is not abelian. (Sections 20.2 and 20.3.)

5. Identify the set of congruence classes of integers modulo n, denoted $\mathbb{Z}_n$. (Section 13.1). Illustrate how addition works in this set by constructing a table of addition facts for $\mathbb{Z}_4$. Show that $\mathbb{Z}_n$ is an abelian group under addition.

6. Considering groups defined abstractly, prove that the identity must be unique. Prove that any element in a group has a unique inverse. Prove the cancellation laws. (Section 20.3.)

7. Suppose that we assume only the existence of a left inverse for each element: for any a, there is a b such that $ba = e$, and that we further assume only that e is a left identity, that $ea = a$ for every a. Prove that b is also a right inverse for a (Hint: there is some left inverse for b. Call it c and consider $cbab$ to show that $ab = e$) Prove that e is also a right identity: that $ae = e$ for all a (Hint: let b be the inverse (now known to be both the left and right inverse) of a and consider the expression bab)