MATHEMATICS 152, FALL 2003
METHODS OF DISCRETE MATHEMATICS
Outline #1 (Counting, symmetry, Platonic solids)

For each item, the class will divide into five groups. Each group will have a different polygon or Platonic solid. Take about ten minutes to carry out the analysis described below, then appoint a member of your group to present it to the class.

1. For your group's Platonic solid (tetrahedron, cube, octahedron, dodecahedron, icosahedron), use counting principles to deduce, given the number of faces, vertices, or edges, the other two quantities.

   Notes on counting:

   - Suppose that set A has m elements and set B has n elements. The Cartesian product $A \times B$ is the set of ordered pairs (a, b), where a is an element of A and b is an element of B. $A \times B$ has mn elements.
   - If set A has n elements, the number of distinct sequences of r different elements (no repetition allowed) is $n(n-1)(n-2)...(n-r+1)$. This can also be written in terms of the factorial function as

     $$\frac{n!}{(n-r)!}$$

   - When counting, it is fine to count things more than once, provided that everything gets overcounted the same number of times and that you know that number.

   Examples

   (a) How many different 3-element subsets can be chosen from a set of 5 elements?

   Answer: There are $5 \times 4 \times 3 = 60$ different 3-element sequences. But this counts each subset 6 times, since the three elements in a subset can be arranged into 6 different orders. So the answer is $\frac{60}{6}$ or 10.

   More generally, the number of different r-element subsets of a set with n elements (sometimes called "n choose r") is

   $$\frac{n!}{(n-r)!r!}$$

   .

(b) How many permutations of 5 elements are there with the cycle structure $(abcde)$? This means that the permutation, viewed as a function, takes $a \longrightarrow b$, $b \longrightarrow c$, $c \longrightarrow d$, $d \longrightarrow e$, $e \longrightarrow a$. Therefore $(abcde)$ means the same thing as $(bcdea)$.

Answer: There are $5! = 120$ sequences of 5 elements. But this counts each permutation 5 times since, for example, the following are all the same:

(1 2 3 4 5), (2 3 4 5 1), (3 4 5 1 2), (4 5 1 2 3), (5 1 2 3 4) So the answer is $\frac{120}{5} = 24$.

(c) An octahedron has 8 faces, each an equilateral triangle. How many edges does it have?

Answer: Each face has 3 edges, and 3 times 8 is 24. But each edge is part of 2 faces, so we counted each edge twice. Divide 24 by 2 to get the answer, 12.

2. For your group's regular polygon (triangle, square, pentagon, hexagon, octagon), enumerate the rotational symmetry operations, specifying the axis and angle of rotation for each.

3. For your group's Platonic solid, enumerate the rotational symmetry operations, specifying the axis and angle for each. An axis of rotation must pass though the center of the solid and can also pass through a vertex, through the center of a face, or through the midpoint of an edge.

4. Associate a permutation with each rotational symmetry operation of your group's regular polygon, and use multiplication of permutations to predict the result of performing successive operations.

   Notes on how to multiply permutations:

   Example 1: Let $a = (2\ 4\ 5)$, $b = (1\ 2\ 3\ 4\ 5)$. Let's calculate ba, which means a followed by b, in accordance with standard function notation.

   Approach 1, the "pure cycle" approach (recommended for the first programming project) Do one element at a time, starting with 1. Determine the result (call it x) of applying a to 1, then the result (call it y) of applying b to x. If y is different from 1, determine next the effect of ba on y. Continue until the result is 1. If the cycle does not include all elements, choose the lowest-numbered one that was omitted, and repeat the process.

   In the given example,

   - 1 is left unchanged by a, then b carries 1 into 2.
   - 2 is carried into 4 by a, then b carries 4 into 5.
   - 5 is carried into 2 by a, then b carries 2 into 3.
   - 3 is left unchanged by a, then b carries 3 into 4.

- 4 is carried into 5 by a, then b carries 5 into 1.

We have found the cycle (1 2 5 3 4). Since it includes every element, we are finished:

Approach 2: The "function list" approach.

If we regard a as a function on the set {1,2,3,4,5}, we have

x	1	2	3	4	5
$a(x)$	1	4	3	5	2

Similarly with b as a function

x	1	2	3	4	5
$b(x)$	2	3	4	5	1

So it is easy to make a table for $b(a(x))$.

x	1	2	3	4	5
$a(x)$	1	4	3	5	2
$b(a(x))$	2	5	4	1	3

Now start with 1 and trace out the cycle (1 2 5 3 4).

Example 2: $a = (1\ 5\ 3)$ $b = (1\ 3\ 4\ 5\ 2)$

The pure cycle approach:

- 1 is carried into 5 by a, then b carries 5 into 2.
- 2 is left unchanged by a, then b carries 2 into 1. So we have found the cycle (1 2).
- Continue with 3, the lowest number that we haven't looked at. 3 is carried into 1 by a, then b carries 1 back into 3. This is a cycle with just one element; conventionally, it is just omitted.
- Continue with 4, the lowest number that we haven't looked at. 4 is left unchanged by a, then b carries 4 into 5.
- 5 is carried into 3 by a, then b carries 3 into 4.

So we have found the cycle (4 5), and the answer is

$ba = (1\ 2)(4\ 5)$.

The function list approach:

x	1	2	3	4	5
$a(x)$	5	2	1	4	3

Similarly with b as a function

x	1	2	3	4	5
$b(x)$	3	1	4	5	2

Make a table for $b(a(x))$.

x	1	2	3	4	5
$a(x)$	1	4	3	5	2
$b(a(x))$	2	1	3	4	5

Start with 1, and find the cycle (1 2).

We still do not know what happens to 3, 4, or 5.

Look at 3 – it is unchanged.

The next number to look at is 4, and we find the cycle (4 5).

So the answer is $ba = (1\ 2)\ (4\ 5)$