MATHEMATICS 152, FALL 2003
METHODS OF DISCRETE MATHEMATICS
Outline #2 (Permutations)

Reading: Biggs, Sections 5.2-5.4, 10.6, 12.5, 12.6, and 21.1. The topic is continued in the rest of chapter 21, but that must wait until you know some group theory.

To gain experience with permutations, play around with groups.exe, which you can download from the course Web site. It is helpful to have at hand your models of the Platonic solids and some regular polygons.

1. (Biggs, sections 5.2 and 10.6) Define "injection," "surjection," and "bijection," and illustrate each concept with a diagram for the case of a function a from a finite set X to a finite set Y. Then define "permutation" and show how to represent as a function the permutation that rearranges 12345 into 32415. Show that there are $n!$ distinct permutations of a set of n elements.

2. (Biggs, section 10.6) Explain how to interpret a permutation expressed in "cycle notation" as a function. Illustrate this by writing (24)(135) as a function from the set {1,2...6} to itself. Then describe an algorithm for converting a permutation that is expressed as a function into cycle notation, and use it to convert

1	2	3	4	5	6	7
4	7	6	2	5	3	1

   which means

   f(1) = 4, f(2) = 7, f(3) = 6,... into cycle notation. (This is very similar to the last topic in outline 1).

3. (Biggs, sections 5.3, 5.4, and 10.6, though part is relegated to the exercises) From the definition of a permutation as a bijection, show that the composition of two permutations is a permutation and that every permutation has an inverse. Show how to calculate the permutation that results from first doing the permutation (123)(45), then the permutation (134). Calculate the permutation that is the result of (134) followed by (123)(45).

4. (Biggs, section 21.1, though Biggs uses only the alibi approach and does not make the issue explicit)

   In representing a rotation of a regular polygon as a permutation (i.e. as a function), there are two equivalent and equally valid approaches. One is the "alibi" approach, which says that the function determines the new location of a vertex in terms of its old location. The other is the "alias" approach, which says that the function determines the new vertex that moves into the location previously occupied by the old one. Demonstrate one rotation of the square for which both approaches lead to the same permutation and one for which the two approaches lead to a different permutation.

5. (Groups.exe and experiment) Start with an icosahedron whose 30 edges are colored with five distinct colors as specified by your "icosahedron kit." Using the "alias" approach (e.g. you rotate the icosahedron and every location where there used to be a red edge is now occupied by a yellow edge, so f(red) = yellow or f(1) = 2) determine the permutation that corresponds to each of the following:
   (a) a rotation about an axis through a pair of vertices.
   (b) a rotation about an axis through the center of a pair of opposite faces.
   (c) a rotation about an axis through the midpoints of two opposite edges.

6. (Biggs, sections 12.5 and 12.6) Write down all the six alternative ways of expressing (245)(31), and state a criterion for selecting one of these six unambiguously. Then count the number of permutations of the five symbols 1,2,3,4,5 with each of the following cycle structures:
   (a) like (123)
   (b) like (12345)
   (c) like (12)(34)

   Show that there is a one-to-one correspondence between these permutations (plus the identity) and the rotations of the regular icosahedron.

7. (Groups.exe and experiment) Show that there is a one-to-one correspondence between all permutations of the set {1,2,3,4} and the set of rotational symmetries of the cube. Identify each the symbols 1, 2, 3, 4 with a pair of vertices of the cube.

8. (Biggs, section 12.6) Express permutations (123) and (12345) as products of transpositions, each in two different ways. Then do the same for (1234) and for (123)(45). Now state and prove Theorem 12.6.1 in Biggs and explain what is meant by an "even" permutation.