MATHEMATICS S-152, SUMMER 2004
The MATHEMATICS OF SYMMETRY

Homework Assignment # 2
Due: Tuesday, July 7, 2005

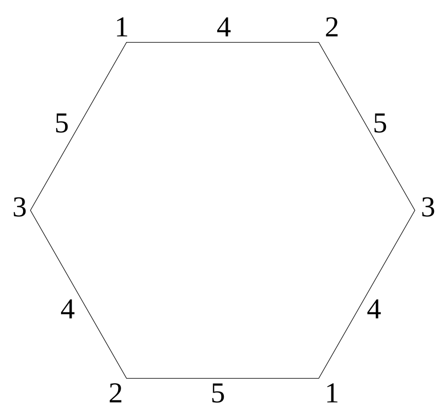
Reading

- Read Biggs, Chapter 20 (Groups)
- Read Biggs, Sections 13.1–13.3 (Modular Arithmetic)

Required Problems

1. In the group S_6, how many cycles are of the form (12)(34)(56)? In the group S_7, how many cycles are of the form (12)(34)(56)(7)?

2. One labeling of the regular hexagon is the figure below, where any symmetry operation on the hexagon permutes the sets of elements that share a number. For example, the three edges numbered 4 may be interchanged with the three edges numbered 5.



   (a) Write the five permutations from S_5 that represent non-trivial rotations about an axis perpendicular to the plane of the hexagon.

   (b) Write the three permutations from S_5 that represent reflections about axes between two opposite vertices (or, if you prefer, 180° rotations about theses axes).

(c) Express the last three non-trivial symmetries of the regular hexagon as permutations from S_5, and describe what each does geometrically.

3. Consider the following six functions:

$$f_1(x) = x \qquad f_2(x) = 1 - x$$
$$f_3(x) = \frac{1}{x} \qquad f_4(x) = \frac{1}{1-x}$$
$$f_5(x) = \frac{x}{x-1} \qquad f_6(x) = \frac{x-1}{x}$$

Now think of these as a group under the operation composition. For example,

$$\begin{aligned}(f_2 \circ f_3)(x) &= f_2(f_3(x)) \\ &= f_2\left(\frac{1}{x}\right) \\ &= 1 - \frac{1}{x} \\ &= \frac{x-1}{x} \\ &= f_6(x).\end{aligned}$$

What group is this?

4. Show that if G is a group and $x, y \in G$, then $(xy)^{-1} = y^{-1}x^{-1}$.

5. In $\mathbb{Z}_{13}$

   (a) What is the additive inverse of [5]?

   (b) For what m and n does $13m + 5n = 1$?

   (c) What is the multiplicative inverse of [5]?

   (d) What is the square root of [-1]?

6. In $\mathbb{Z}_{12}$

(a) Which elements have no multiplicative inverse?

(b) What is the multiplicative inverse of each of the remaining elements?

(c) Prove that the invertible elements form a group, and write out the multiplication table for this group.

7. Given a set G with a binary operation (denoted by adjacency of elements) that is closed under the operation and that satisfies the three following axioms:

   G1'. Given any $x, y, z \in G$, $(xy)z = x(yz)$.

   G2'. Given any $a, b \in G$, there is a unique element $x \in G$ such that $xa = b$.

   G3'. Given any $a, b \in G$, there is a unique element $y \in G$ such that $ay = b$.

   Show that G is a group under this operation.

Exploratory Problems

1. Describe the symmetry group of the circle.

2. Given a set S with a binary operation (denoted by adjacency of elements) that is both closed and associative, suppose we have the following axiom in place of our usual identity axiom:

   S3.' Given $x \in S$, there is an element $e \in S$ (possibly depending on x) such that $xe = ex = x$.

   For example, S could be the set of 2×2 matrices of the form $\left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right]$with $ab = 0$. Then if x is the matrix $\left[\begin{smallmatrix} 0 & 0 \\ 0 & b \end{smallmatrix}\right]$ e can be the matrix $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$.

   There is no inverse axiom, since we are not necessarilyconsidering a group.

   Decide whether or not the usual identity axiom holds. Either prove that it does, or invent a counterexample.

10. Biggs, Exercises 20.4, number 3. "Show that M is a group" means "show that all the group axioms are satisfied."

3. Consider the set $\mathbb{Q}^\times$ of non-zero rational numbers, and define the binary operation $*$ as follows,

$$x * y = \begin{cases} xy, & \text{if } x, y > 0 \\ xy, & \text{if } xy < 0 \\ 2xy, & \text{if } x, y < 0 \end{cases}$$

That is, if x and y are both positive, then $x * y$ is the usual product, and if one of x and y is positive, then $x * y$ is also the usual product, but if both x and y are negative, then $x * y$ is twice the usual product.

Verify the four group axioms for $(\mathbb{Q}^\times, *)$.