MATHEMATICS S-152, SUMMER 2005
The MATHEMATICS OF SYMMETRY

Homework Assignment # 3
Due: Thursday, July 14, 2005

Reading

- Biggs, Chapter 20, especially 20.4–20.9.

Note. The quiz on Thursday, July 14 will cover all material on this problem set but not the topic of quotient groups (the last two topics of outline 5).

Required Problems

1. Write out the group multiplication table for the invertible elements of $\mathbb{Z}_{18}$ under multiplication.

2. In class or in the preceding problem, we have discussed the following seven groups, all of which have six elements.

   (a) The symmetry group of the equilateral triangle, D_3

   (b) The permutation group on the set of three letters, S_3

   (c) The group of additive congruences modulo 6, $(\mathbb{Z}_6, \oplus)$

   (d) The multiplicative group of non-zero congruences modulo 7, $(\mathbb{Z}_7^\times, \otimes)$

   (e) The six rotations of a regular hexagon about an axis perpendicular to the hexagon, C_6

   (f) The six functions from HW #2, problem 3 under composition.

   (g) The invertible elements of $\mathbb{Z}_{18}$ under multiplication.

   Are these different manifestations of the same group? Which of them are abelian? Which are cyclic? Find a natural identification between the ones that are the "same."

3. Show that [3] is a generator for the multiplicative group $\mathbb{Z}_7^\times$, and find any other generators. (For the definition of generator, see Biggs, Section 20.6.)

4. Find subgroups of the symmetry group of the icosahedron isomorphic to D_3, D_5, and A_5, and for each, find two permutations that generate the subgroup. (You can check your answer using groups.exe, but find different subgroups than the ones given by clicking the "Show ..." buttons in the program.)

5. Find the order of every element of $\mathbb{Z}_{13}^{\times}$. In particular, show that $\mathbb{Z}_{13}^{\times}$ is cyclic, and find all of its generators.

6. Find all subgroups of D_3 and D_4.

7. Let $G = D_4$, where one rotation by $\frac{\pi}{2}$ is denoted a and one reflection is denoted b, so that we may write $G = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$. Let $H_1 = \{e, b\}$ be the subgroup consisting of the identity and one reflection, and let $H_2 = \{e, a^2\}$ be the subgroup consisting of the identity and the rotation by π. Find all left-cosets and all right-cosets of H_1 and H_2.

8. Show how we may consider S_3 to be a subgroup of S_4, and find all of its left-cosets. Give a geometric interpretation of this in terms of the cube and the equilateral triangle.

9. For the symmetry groups of the Platonic solids, list all possible orders of subgroups, and for any such subgroup, indicate the number of elements in a left-coset and the number of left-cosets.

10. Let $G = A_5$, and consider the subgroup $H_1 = \langle(123)\rangle$ (the subgroup generated by (123)).

    (a) Find the subgroup of G conjugate to H_1 under the element $g = (14)(25)$.

    (b) Find the subgroup of G conjugate to H_1 under the element $g = (12)(45)$.

    (c) Is H_1 self-conjugate? Explain.

Exploratory Problems

11. Let G be a finite group (that is, $|G| < \infty$), and let $g \in G$. Prove that the order of g is finite.

12. Find all generators of the cyclic group C_n. *Hint*: Try the case when n is prime first.

13. Let G be a finite group, and suppose that H is a subgroup such that $|G| = 2|H|$. Show that every left-coset of H is also a right-coset of H.