MATHEMATICS S-152, SUMMER 2005
The MATHEMATICS OF SYMMETRY

Homework Assignment # 4
Due: Tuesday, July 19, 2004

Reading

- Read Biggs, Chapter 22.

Required Problems

1. Show that $H = \{I, (12)\}$ is not a normal subgroup of S_4 by computing its conjugate subgroups.

2. Show that $J = \{I, (123), (132)\}$ is a normal subgroup of S_3 by computing its conjugate subgroups. What is the quotient group S_3/J?

3. Consider the group $G = (\mathbb{Z}_{13}^{\times}, \otimes)$ and the subgroup H generated by [4]. Determine the quotient group G/H by writing down its elements (the cosets) and writing a group table for them.

4. Consider the ring $R = M_2(\mathbb{Z}_2)$, that is, the 2×2 matrices with entries from the field $\mathbb{Z}_2$.

   (a) How many elements does R have?

   (b) Find all elements that have multiplicative inverses, and list them with their inverses.

5. Consider the ring of *Gaussian integers*, $R = \mathbb{Z}[i] = \{a + bi | a, b \in \mathbb{Z}\}$, where $i^2 = -1$. With the usual addition and multiplication from the complex numbers, it may be shown that R is a ring. Which elements have multiplicative inverses, and what are they?

6. Problem #22.1.3 in Biggs: Show that if x and y are members of a ring R then $(-x)(y) = -(xy)$ and $(-x)(-y) = xy$. At each stage of the proof, explain which property of R you are using.

Exploratory Problems

7. In this exercise, we consider the group A_4 and its normal subgroup V_4, the Klein Four group, and show that the quotient group A_4/V_4 is isomorphic to C_3.

   Exhibit V_4 as a (normal) subgroup of the rotation group of the tetrahedron. List as permutations the members of the three cosets. Identify each coset with an element of C_3. For each pair of cosets (recall that C_3 is abelian!), perform the multiplication by members of the cosets chosen at random, and thereby reconstruct the multiplication table for C_3.