MATHEMATICS S-152, SUMMER 2005
The MATHEMATICS OF SYMMETRY
Homework Assignment # 6
Due: August 2, 2005

Reading

Finish the handout on Affine Geometry.

Required Problems

1. In the large affine faculty senate two "triangles" are Danny-Gavin-Viola (ABC) and Helen-Sally-Xenia ($A'B'C'$). Show that these six instructors satisfy the conditions and the conclusion of the Desargues axiom.

2. In the medium affine faculty senate, choose the library committee, with Jane as the additive identity. Add Greg and Mike, first using Kate as the auxiliary point, then using Irma. Draw a single diagram representing the addition with both auxiliary instructors, attaching a committee name to each line and an instructor name to each point. Identify the two "triangles" that are related by Desargues' theorem in the proof that the answer is independent of the choice of instructor.

3. Use the data for the medium affine faculty senate. Choose the library committee, with Jane as the additive identity and Dave as the multiplicative identity. Multiply Greg by Mike, then multiply Mike by Greg, in each case using Kate as the auxiliary point. Draw a single diagram representing the multiplication in both orders, attaching a committee name to each line and an instructor name to each point. Identify the degenerate hexagon of six instructors to which Pappus' Theorem (axiom A5) must be applied to show that the multiplication is commutative.

4. Consider the yearbook committee in the large affine faculty senate. Choose Helen as the additive identity 0 and Sally as the multiplicative identity I. Generate the addition and multiplication tables using affine.exe, and use them to identify the members of the committee with the elements 0, 1, 2, 3, 4 of $\mathbb{Z}_5$.

5. Suppose that A and E are members of a committee of an affine faculty senate whose additive identity is O. Describe a method for constructing the instructor C such that $A + C = E$ using auxiliary instructor B. Prove directly from the axioms that using a different instructor B' on the same committee as O and B leads to the same C. (Of course, this is true for arbitrary B', but the proof becomes tedious). Include a diagram to illustrate how your construction would look in the Euclidean plane, for which your proof is also valid.

6. Suppose that C and E are members of a committee of an affine faculty senate whose additive identity is O and whose multiplicative identity is I. Describe a method for constructing the instructor A such that $AC = E$ using auxiliary member B. Prove directly from the axioms that using a different member B' on the same committee as I and B leads to the same A. (Of course, this is true for arbitrary B', but the proof becomes tedious). Include a diagram to illustrate how your construction would look in the Euclidean plane, for which your proof is also valid.

7. Choose the quality committee in the large affine faculty senate, with Betty as the additive identity O, Isaac as A, Kevin as C, Yoric as E. Choose James as auxiliary instructor B. Carry out the addition $(A + C) + E$ and $A + (C + E)$, using auxiliary instructors B' and D exactly as described in the notes and in the diagrams on the Web.

   (a) Who is B', and who is D?

   (b) Who is the sum $A + C + E$?

   (c) Redraw the diagram from the Web that shows that addition is associative, with names attached to all the instructors and committees. If two distinct points in the diagram have the same name, don't worry—the diagram is for Euclidean geometry and you are doing finite geometry.

8. Consider the large affine faculty senate. Choose the housing committee as the x axis and the overseers committee as the y axis. Maria, who serves on both these committees, has coordinates $(0, 0)$. Let Wally be the multiplicative identity on the housing committee so that his coordinates are $(1, 0)$. Let Emily be the multiplicative identity on the

overseers committee so that her coordinates are $(0,1)$. Every instructor now has a coordinate pair (x,y), where x and y are integers mod 5, represented by 0, 1, 2, 3, and 4. The easiest way to find these coordinates for an "axis" committee is to start with the multiplicative identity 1 and keep adding on the multiplicative identity to get 2, 3, and 4. The coordinates of an instructor are found simply by looking at the committees on which that instructor serves that are parallel to the two coordinate axes and taking the values associated with the intersections of these committees with the axes.

(a) What instructor has coordinates $(1,2)$?

(b) What are the coordinates of Betty in this coordinate system?

(c) What committee satisfies the equation $y = 3x + 2$?

(d) What equation is satisfied by the diversity committee?

9. Consider the medium affine faculty senate. Choose Greg as the origin. Choose the quality committee as the x-axis, with Lynn as multiplicative identity. Choose the budget committee as y-axis, with Emma as multiplicative identity. The finite field $\mathbb{F}_4$ has elements $[0],[1],[u],[u+1]$, where $[u][u+1] = [1]$, and all arithmetic is carried out modulo 2; i.e. $[1] + [1] = [0]$.

   (a) What instructor has coordinates $([1],[1])$? What committee satisfies the equation $y = x$? Assign coordinates $([u],[u])$ to whichever remaining instructor on this committee is first alphabetically.

   (b) Determine the coordinates of all the instructors. Show your answer on a diagram, with the x axis horizontal and the y axis vertical.

   (c) Which committee satisfies $y = [u]x$, and which satisfies $y = [u+1]x$?

   (d) What equation is satisfied by the insurance committee?

   (e) What committee satisfies the equation $x + [u]y = [u+1]$?

10. Consider the matrix $\left[\begin{smallmatrix} 1 & 0 \\ 3 & 1 \end{smallmatrix}\right]$ from $SL(2,\mathbb{Z}_5)$. Let this matrix act five times in succession, starting with the vector $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$, and thereby verify that it permutes the five vectors of the form $\left[\begin{smallmatrix} 1 \\ y \end{smallmatrix}\right]$.

Exploratory Problems

Some of these refer to Bennett, *Affine and Projective Geometry*, on reserve on Cabot Library or in the Birkhoff Math Library on the 3rd floor of the Science Center (use Hollis to get the call number – there is no card in the card catalog).

11. Consider the kitchen committee in the large affine senate, with Alice as additive identity O and Ralph as multiplicative identity I. Let James be point A in Figure 3.20 on p. 67 of Bennett; let Nancy be C and Viola E. Choose Xenia as the auxiliary instructor B. Redraw Figure 3.20 from Bennett (for the proof of the distributive law $(A + C)E = AE+CE$), attaching an instructor name to each point and a committee name to every line.

12. Bennett, problem 7 on page 55.

13. Let G be the group of symmetries of the cube. (Recall that $|G| = 24$.) By orienting the cube in $\mathbb{R}^3$ with its vertices at the eight points $(\pm 1, \pm 1, \pm 1)$, write down the 3×3 matrices that represent these symmetries.

    Check your work by verifying that the numbers of matrices of orders 1,2,3, and 4 are 1,6,8, and 9, respectively. What are the eigenvalues for one matrix of each type, and what are the multiplicities?