MATHEMATICS S-152, SUMMER 2005
The MATHEMATICS OF SYMMETRY
Homework Assignment # 7
Due: August 9, 2005

Required Problems

1. Prove by brute force the Cayley-Hamilton theorem for 2×2 matrices: a matrix A satisfies its own characteristic equation, or

$$A^2 - (\mathrm{tr}A)A + (\det A)I = 0.$$

2. How many matrices are in the group $GL(3, \mathbb{Z}_3)$ and how many in $SL(3, \mathbb{Z}_3)$? In each case, how many of the matrices are diagonal?

3. A collineation of an affine faculty senate is a bijection f that maps instructors to instructors and committees into committees in such a way that

   - If instructors A and B serve on committee s, then $f(A)$ and $f(B)$ both serve on $f(s)$.
   - If committees s and t intersect in instructor A (or are parallel), then $f(s)$ and $f(t)$ intersect in $f(A)$ (or are parallel).

   For the small affine faculty senate, any collineation that fixes three points (instructors) is the identity and every collineation is an affine transformation, a linear transformation (represented by a matrix A) followed by a translation (addition of a vector $\vec{b}$).

   (a) Find the unique collineation that maps Bob into Ian, Hal into Cal, and Gus into Amy. Make a table showing the image of each instructor and each committee.

   (b) Find a matrix A and vector $\vec{b}$ that represents this mapping in a coordinate system where Bob is (0,0), Hal is (1,0), and Gus is (0,1). The entries in the matrix and vector will be integers modulo 2, so they will all be 0, 1, or 2.

4. For the medium affine senate, there are collineations that hold three points (instructors) fixed but that are not the identity. Find such a collineation that holds Dave, Jane, and Owen fixed. Make a table showing the image of each instructor and each committee. Since the square of this collineation is the identity, it either holds a given committee or instructor fixed or interchanges two committees or instructors; so your table only has to show the instructors and committees that get interchanged.

   Hint: Choose one instructor (say Dave) as the additive identity for both axes, the other two (Jane and Owen) as the multiplicative identities for the two axes. Find the instructor whose coordinates are $(1, 1)$. Now you have an arbitrary choice about who on the committee containing $(0, 0)$ and $(1, 1)$ has coordinates (u, u) and who has $(u + 1, u + 1)$. Interchanging these two instructors leads to the desired collineation.

5. A much smaller group than $SL(2, \mathbb{F}_4)$ is $SL(2, \mathbb{Z}_2)$. Write down the six matrices of this group, and show that it is isomorphic to S_3.

6. Find a 6-element subgroup of $SL(2, \mathbb{F}_4)$ that includes

$$\begin{bmatrix} x & 1 \\ 0 & x+1 \end{bmatrix}$$

   and is isomorphic to S_3, the group of symmetries of the equilateral triangle.

   Hint: One way to approach this is to use the diagram attached to the notes. The given matrix is a rotation about one of the vertices of the top pentagon, involving the three vectors, $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 1 \\ x+1 \end{smallmatrix}\right]$, and $\left[\begin{smallmatrix} 1 \\ x \end{smallmatrix}\right]$. You need the three elements of order 2 to complete the subgroup.)

7. In the two preceding problems, you found two conjugate, isomorphic subgroups of $SL(2, \mathbb{F}_4)$. Find a matrix B with the property that for any matrix A in $SL(2, \mathbb{Z}_2)$ (problem 5), the matrix BAB^{-1} is the corresponding element of the subgroup that you found in problem 6.

   Hint: matrix B carries one axis of rotation into the other.

Exploratory Problems

8. Let $A \in M_n(F)$. Recall that the *characteristic polynomial* of A is $f_A(\lambda) = \det(A - \lambda I)$, and that λ is an eigenvalue of A if and only if

$f_A(\lambda) = 0$. Define the *algebraic multiplicity* of the eigenvalue λ_0 to be the number of times λ_0 was a root of the characteristic polynomial. (That is, if k is the algebraic multiplicity, we may write $f_A(\lambda) = (\lambda - \lambda_0)^k \cdot g(\lambda)$, where $g(\lambda_0) \neq 0$.) Define the *geometric multiplicity* of λ_0 to be the dimension of its corresponding eigenspace, $dim(E_{\lambda_0})$. Show that the algebraic multiplicity of an eigenvalue is always greater than or equal to its geometric multiplicity.

9. Find a 10-element subgroup of $SL(2, \mathbb{F}_4)$ that includes

$$\begin{bmatrix} 0 & 1 \\ 1 & x \end{bmatrix}$$

and is isomorphic to D_5, the group of symmetries of the regular pentagon.

Hint: Consider the 180° rotation about an axis in the equatorial plane that carries $\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]$ into itself. You know what this does to $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$, so you can write down the matrix. That matrix, along with $\left[\begin{smallmatrix} 0 & 1 \\ 1 & x \end{smallmatrix}\right]$ generates the group.)

10. Conway's atlas of groups claims that $SL(3, \mathbb{Z}_2)$ has 168 elements, some of which are of order 7. Confirm the number of elements, find one element whose order is 7, and compute its powers.