MATHEMATICS S-152, SUMMER 2005
The MATHEMATICS OF SYMMETRY
Homework Assignment # 8
Due: August 11, 2005

Required Problems

1. Use the Cayley-Hamilton theorem to show that in $SL(2, \mathbb{F}_4)$
   (a) a matrix A with trace 0 has order 2
   (b) a matrix A with trace 1 has order 3
   (c) a matrix A with trace x has order 5

2. Find a subgroup of 6 matrices in $PSL(2, \mathbb{Z}_5)$ that is isomorphic to S_3. Explicitly identify each matrix in the subgroup with a permutation such as (12) or (123).

3. Find a subgroup of 12 matrices in $SL(2, \mathbb{F}_4)$ that is isomorphic to A_4. Explicitly identify each matrix in the subgroup with a permutation such as (12)(34) or (123).

   Hint: Each matrix has the same eigenvector.

4. With the two-component vectors over $\mathbb{F}_4$ arranged into lines as in the notes, i.e.

   - Line 1: multiples of $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$
   - Line 2: multiples of $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$
   - Line 3: multiples of $\left[\begin{smallmatrix}1\\x\end{smallmatrix}\right]$
   - Line 4: multiples of $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$
   - Line 5: multiples of $\left[\begin{smallmatrix}1\\x+1\end{smallmatrix}\right]$

   Determine the action of each of the following matrices from the group $SL(2, \mathbb{F}_4)$ on the lines, and hence associate a permutation with the matrix. You will learn more if you start by determining the eigenvalues and eigenvectors, but you can also just use a brute-force approach and let the matrix operate on a vector from each of the five lines.

(a)

$$\begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

(c)

$$\begin{bmatrix} x+1 & 1 \\ 0 & x \end{bmatrix}$$

5. With the two-component vectors over $\mathbb{F}_4$ arranged into lines as in the preceding problem, construct the matrix of $SL(2, \mathbb{F}_4)$ that represents each of the following permutations:

   (a) (13)(25)

   (b) (13254)

   (c) (132)

   Hint: the first and second columns of the matrix in each case are multiples a and b of some vectors on the lines to which lines 1 and 2 respectively are mapped. Use a piece of information about the other lines to set up an equation relating the multiple a for the first column to the multiple b for the second, then choose a and b to make the determinant 1.

6. With the two-component vectors over $\mathbb{Z}_5$ arranged into lines as in the notes, i.e.

   - Line 1: multiples of $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$
   - Line 2: multiples of $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$
   - Line 3: multiples of $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$
   - Line 4: multiples of $\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$
   - Line 5: multiples of $\left[\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right]$ if you prefer
   - Line 6: multiples of $\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix} 1 \\ 4 \end{smallmatrix}\right]$ if you prefer.

   Determine the action of each of the following matrices from the group $PSL(2, \mathbb{Z}_5)$ on the lines, and hence associate a permutation with the matrix.

   You will learn more if you start by determining the eigenvalues and eigenvectors, but you can also just use a brute-force approach and let the matrix operate on a vector from each of the six lines.

(a) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

7. With the two-component vectors over $\mathbb{Z}_5$ arranged into lines as in the preceding problem, let $G = PSL_2(\mathbb{Z}_5)$ be the group acting on V by permuting the lines.

   (a) Write down the four matrices in $SL_2(\mathbb{Z}_5)$ that take Line 2 to Line 6 and Line 1 to Line 4.

   (b) Which two of these matrices take Line 5 to itself? Write down the complete permutation (in S_6) that corresponds to these matrices. To what line do the other two matrices take Line 5?

   (c) Using a generalization of the technique from parts (a) and (b), invent a counting argument to show that $|G| = |A_5|$.

8. A much smaller group than $SL(2, \mathbb{Z}_5)$ is $SL(2, \mathbb{Z}_3)$. For this group, determine how many elements of order 1, 2, and 3 there are. Show that if you pair up matrices that differ only by an overall sign, you get a group that is isomorphic to the group of symmetries of a regular tetrahedron.

9. (a) Exhibit subgroups of A_5 and of A_6 that are both isomorphic to D_5 and so to one another. (If you use groups.exe, this will be very easy!)

   (b) Exhibit subgroups of $SL(2, \mathbb{F}_4)$ and $PSL(2, \mathbb{Z}_5)$ that are both isomorphic to D_5 and so to one another.

   Hint: The corresponding permutations that you found in part a tell you how each matrix permutes the lines of vectors.

Exploratory Problems

10. Find a 12-element subgroup of $SL(2, \mathbb{F}_4)$ that includes

$$\begin{bmatrix} x & x+1 \\ x+1 & x \end{bmatrix}.$$

Hint: This matrix has one eigenvector, and there are 11 others with the same eigenvector: the identity, two more of order 2, and eight of order 3.

11. Using the same approach as in the notes, determine how many elements are in the group $SL(2, \mathbb{Z}_7)$, then show that the group $PSL(2, \mathbb{Z}_7)$ has 168 elements. Determine how many of these elements have order 7.

12. Conway's atlas of groups claims that $PSL(2, \mathbb{F}_9)$ has 360 elements and is isomorphic to A_6. Show that each of these groups has 360 elements, and show that each group has the same number of elements of each order.