MATHEMATICS S-152, SUMMER 2005
THE MATHEMATICS OF SYMMETRY

Outline #1 (Counting, symmetry, Platonic solids, permutations)

The class will divide into four groups. Each group will have a different polygon or Platonic solid. Please take five to ten minutes to carry out each of the six numbered exercises described below, then appoint a member of your group to present it to the class.

1. For your group's Platonic solid (tetrahedron, cube, dodecahedron, icosahedron), use counting principles to deduce, given the number of faces, vertices, or edges, the other two quantities.

Notes on Counting

- Suppose that set A has m elements and set B has n elements. The Cartesian product $A \times B$ is the set of ordered pairs (a, b), where a is an element of A and b is an element of B. The set $A \times B$ has mn elements.
- If set A has n elements, then the number of distinct sequences of r different elements (no repetition allowed) is

$$n(n-1)(n-2)...(n-r+1) = \frac{n!}{(n-r)!}.$$

- When counting, it is fine to count things more than once, provided that everything gets over counted the same number of times and that you know that number.

Examples

(a) *Question*: How many different 3-element subsets can be chosen from a set of 5 elements?

Answer: There are $5 \times 4 \times 3 = 60$ different 3-element sequences. But this counts each subset 6 times, since the three elements in a subset can be arranged into 6 different orders. So the answer is $60/6 = 10$.

More generally, the number of different r-element subsets of a set with n elements (sometimes called "n choose r") is

$$\frac{n!}{(n-r)!r!}.$$

(b) *Question*: How many permutations of 5 elements are there with the cycle structure $(abcde)$? This means that the permutation, viewed as a function, takes $a \longrightarrow b$, $b \longrightarrow c$, $c \longrightarrow d$, $d \longrightarrow e$, $e \longrightarrow a$. Therefore $(abcde)$ means the same thing as $(bcdea)$.

Answer: There are $5! = 120$ sequences of 5 elements. But this counts each permutation 5 times since, for example, the following are all the same:

$$(12345), (23451), (34512), (45123), (51234)$$

So the answer is $120/5 = 24$.

(c) *Question*: An octahedron has 8 faces, each an equilateral triangle. How many edges does it have?

Answer: Each face has 3 edges, and 3 times 8 is 24. But each edge is part of 2 faces, so we have counted each edge twice. Divide 24 by 2 to get the answer, 12.

(d) *Question*: An octahedron has 6 vertices. Four faces meet at each vertex. How many faces does it have?

Answer: There are 6 vertices, and 4 times 6 is 24. But each face includes 3 vertices, so we have counted each face three times. Divide 24 by 3 to get the answer, 8.

2. For your group's regular polygon (triangle, square, pentagon, hexagon), enumerate the rotational symmetry operations, specifying the axis and angle of rotation for each.

3. For your group's Platonic solid, enumerate the rotational symmetry operations, specifying the axis and angle for each. An axis of rotation must pass though the center of the solid and can also pass through

a vertex, through the center of a face, or through the midpoint of an edge.

For the equilateral triangle, the permutation $a = (123)$ represents a 120-degree rotation about an axis perpendicular to the triangle, while the permutation $b = (23)$ represents a 180-degree rotation (or a reflection) that interchanges vertices 2 and 3. The product ba (a followed by b) of these two permutations, (23)(123), is equal to (13). Why?

- (123) takes 1 to 2, then (23) takes 2 to 3.
- (123) takes 2 to 3, then (23) takes 3 back to 2.
- (123) takes 3 to 1, then (23) leaves 1 alone.
- So the net effect of ba is to interchange 1 and 3.

Similarly, the product (13)(12), is equal to (123). Why?

- (12) takes 1 to 2, then (13) leaves 2 alone.
- (12) takes 2 to 1, then (13) takes 1 to 3.
- (12) leaves 3 alone, then (13) takes 3 to 1, .
- So the net effect is $1 \to 2 \to 3 \to 1$.

4. Each group should illustrate one of these results by carrying out rotations of a triangle. There are two equally good ways to do this, described in the notes below.

   - Group 1: (23)(123) = (13) using the alibi approach, with axes fixed on the blackboard (the most natural approach).
   - Group 2: (23)(123) = (13) using the alias approach, with axes fixed in the triangle (probably the best all-purpose approach).
   - Group 3: (13)(12) = (123) using the alibi approach, with axes fixed on the blackboard (the most natural approach).
   - Group 4: (13)(12) = (123) using the alias approach, with axes fixed in the triangle (probably the best all-purpose approach).

Note on "Alias" Versus "Alibi"

Suppose that you have a pentagon whose vertices are adjacent to the numbers 1 through 5 on the blackboard. One way to associate a permutation (a function) with a given rotation is to carry out the rotation and see "where the vertex that starts at 1 goes". This is the "alibi" approach, and it is perfectly OK. This approach works well for planar figures like polygons. This method is difficult to use for Platonic solids because you have to keep track of positions that are fixed in three-dimensional space.

Somewhat simpler is the "alias" approach. Now, the numbers are attached to the vertices of the pentagon, so that they move around on the blackboard as the pentagon is rotated. Concentrate on the position occupied by vertex 1, and see what new vertex moves in to occupy that position. That is easy to do. It leads to a different permutation from the alibi approach (the inverse permutation, to be precise), and it is neither better nor worse in principle—just different. For Platonic solids this approach is much easier.

You must not mix the two approaches—choose one, and stick to it.

Note on "Space Axes" Versus "Body Axes"

Another ambiguity in associating permutations with symmetry operations concerns "axes fixed in the body" versus "axes fixed in space." Either is okay.

Suppose you are rotating a square whose vertices are adjacent to the numbers 1 through 4 on the blackboard. Then (24) might refer to "a rotation about the line that runs northeast to southwest." The axis passes through the fixed positions 1 and 3, and the rotation interchanges the vertices that are next to positions 2 and 4. This approach combines naturally with the alibi approach, and it works well for polygons.

Suppose, instead you are rotating a square whose vertices are numbered 1 through 4. Then (24) might refer to a rotation about the line that runs through vertices 1 and 3, which interchanges the vertices numbered 2 and 4. The axis of rotation will not always coincide with the same line on the blackboard. This approach combines naturally with the alias approach, and it works well for Platonic solids.

You must not mix the two approaches—choose one, and stick to it.

When you use the alibi approach, successive permutations describe the journey of one specific vertex. The symbols in the permutation refer to fixed positions on the paper or blackboard. So (23)(12) means, "Choose the vertex that starts next to position 1. Carry out a rotation that swaps the vertices next to positions 1 and 2, then carry out a rotation that swaps the vertices next to positions 2 and 3." The chosen vertex will move to position 2, then to position 3.

When you use the alias approach, successive permutations describe the vertices that occupy one specific position. The symbols in the permutation refer to fixed vertices of the polygon or Platonic solid. So (23)(12) means, "Choose the position initially occupied by vertex 1. Carry out a rotation that swaps the vertices numbered 1 and 2, then carry out a rotation that swaps the vertices numbered 2 and 3." The chosen position will get occupied by vertex 2, then by vertex 3.

Summary

- *With the alibi approach, use axes fixed in space.*
- *With the alias approach, use axes fixed in the body.*

5. Associate a permutation with two rotational symmetry operations of your group's regular polygon, and demonstrate an example of using multiplication of permutations to predict the result of performing successive operations. That is, identify two symmetry operations a and b, associate a permutation with each, and show that the product ba (a followed by b) correctly predicts the result of performing operation a followed by operation b. For the sake of uniformity, use the alibi approach and axes fixed in space. That means that when you present your example, you will have to copy numbers from the vertices of your polygon onto the blackboard.

   To keep things interesting, at least one of the symmetry operations, and perhaps both, should be a 180-degree rotation about an axis in the plane of the polygon.

Notes on How to Multiply Permutations

Example 1 : Let $a = (245)$, $b = (12345)$. Let us calculate ba, which means a followed by b, in accordance with standard function notation.

- *Approach 1*, the "pure cycle" approach (recommended for the first programming project) Do one element at a time, starting with 1. Determine the result (call it x) of applying a to 1, then the result (call it y) of applying b to x. If y is different from 1, determine next the effect of ba on y. Continue until the result is 1. If the resulting cycle does not include all elements, choose the lowest-numbered one that was omitted, and repeat the process.

  In the given example,

  - 1 is left unchanged by a, then b carries 1 into 2.
  - 2 is carried into 4 by a, then b carries 4 into 5.
  - 5 is carried into 2 by a, then b carries 2 into 3.
  - 3 is left unchanged by a, then b carries 3 into 4.
  - 4 is carried into 5 by a, then b carries 5 into 1.

  We have found the cycle (12534). Since it includes every element, we are finished:

- *Approach 2*: The "function list" approach.

  If we regard a as a function on the set $\{1, 2, 3, 4, 5\}$, we have

$$\begin{pmatrix} x & 1 & 2 & 3 & 4 & 5 \\ a(x) & 1 & 4 & 3 & 5 & 2 \end{pmatrix}$$

  Similarly with b as a function

$$\begin{pmatrix} x & 1 & 2 & 3 & 4 & 5 \\ b(x) & 2 & 3 & 4 & 5 & 1 \end{pmatrix}$$

  So it is easy to make a table for $b(a(x))$.

$$\begin{pmatrix} x & 1 & 2 & 3 & 4 & 5 \\ a(x) & 1 & 4 & 3 & 5 & 2 \\ b(a(x)) & 2 & 5 & 4 & 1 & 3 \end{pmatrix}$$

  Now start with 1 and trace out the cycle (1 2 5 3 4).

Example 2 : $a = (1\ 5\ 3)$ $b = (1\ 3\ 4\ 5\ 2)$

- *The pure cycle approach*:
  - 1 is carried into 5 by a, then b carries 5 into 2.
  - 2 is left unchanged by a, then b carries 2 into 1. So we have found the cycle (1 2).
  - Continue with 3, the lowest number that we have not examined. 3 is carried into 1 by a, then b carries 1 back into 3. This is a cycle with just one element; conventionally, it is just omitted.
  - Continue with 4, the lowest number that we have not considered. 4 is left unchanged by a, then b carries 4 into 5.
  - 5 is carried into 3 by a, then b carries 3 into 4.

  So we have found the cycle (4 5), and the answer is
  $ba = (1\ 2)(4\ 5)$.

- *The function list approach*:

$$\begin{pmatrix} x & 1 & 2 & 3 & 4 & 5 \\ a(x) & 5 & 2 & 1 & 4 & 3 \end{pmatrix}$$

  Similarly with b as a function

$$\begin{pmatrix} x & 1 & 2 & 3 & 4 & 5 \\ b(x) & 3 & 1 & 4 & 5 & 2 \end{pmatrix}$$

  Make a table for $b(a(x))$.

$$\begin{pmatrix} x & 1 & 2 & 3 & 4 & 5 \\ a(x) & 1 & 4 & 3 & 5 & 2 \\ b(a(x)) & 2 & 1 & 3 & 4 & 5 \end{pmatrix}$$

  Start with 1, and find the cycle (1 2).
  We still do not know what happens to 3, 4, or 5.
  Look at 3 – it is unchanged.
  The next number to look at is 4, and we find the cycle (4 5).
  So the answer is $ba = (1\ 2)\ (4\ 5)$

6. For additional practice in counting, consider various types of permutations of the symbols 1 through 5, and count how many distinct permutations there are with a given cycle structure. Here are all the possibilities:

   (a) the identity

   (b) permutations like (12)

   (c) permutations like (123)

   (d) permutations like (1234) (hard)

   (e) permutations like (12345)

   (f) permutations like (12)(34) (hard)

   (g) permutations like (12)(345)

   Choose three at random to do, including one of the hard ones. Then we will compare notes. We will know that something is wrong if the sum is not 120, since each of the 120 permutations of the symbols 1 through 5 falls into one of the categories.

   One way do do this is to consider all the decisions that you would have to make in order to specify a permutation completely. You will then have to correct for over counting because, for example, (12) and (21) are the same permutation.

   Given that you have to correct for over counting in any event, perhaps the easiest approach is to write out the cycles that have just a single element, so that for (123) you would have a pattern like $(xxx)(x)(x)$. There are now in every case $5! = 120$ different ways to replace the 5 x's with the symbols 1 through 5. This over counts the number of permutations, for two reasons:

   - For a cycle with k elements, there are k different ways to write it, but they all describe the same permutation.
   - If there are j cycles, each with k elements, there are $j!$ different orders for these cycles, but they all describe the same permutation.