MATHEMATICS S-152, SUMMER 2005
THE MATHEMATICS OF SYMMETRY

Outline #12 (Graphs of Groups)

References

This is based on the attached excerpts from Grossman and Magnus, *Groups and Their Graphs*. Most of the rest of this book covers material that is already familiar. If you want to read more, there is one copy in Cabot Library and one in the Math library on the 3rd floor of the Science Center. The last three topics are somewhat challenging because Grossman and Magnus do not provide the details!

1. Draw the graph for a finite cyclic group such as C_4, C_5, or C_6, which has a single generator a. Explain the connection between the group and its graph, with particular attention what features of the graph correspond to the identity I and to the element a^{-1}. Give an example of how different "words", corresponding to different "walks," (the authors use "path" but we will stick to Biggs' terminology) can represent the same group element. (pages 44–47)

2. Draw the graph for the symmetry group of the equilateral triangle, with generators r (a "rotation") and f (a "flip"). Give a nontrivial example of an "empty word" and give examples of cycles in this graph that represent "relations". Explain the convention whereby an edge without an arrow represents a generator of order 2. (pages 48–55)

3. Define the following terms and give an example of each for the cyclic group C_3.

   (a) "relation" or "generator relation"

   (b) "defining relation"

   (c) "equivalent words"

   (d) "class of equivalent words"

   Show that there are three equivalence classes of words for C_3 (pages 56–63)

4. Show that the defining relations for D_3,

   (a) $r^3 = I$,

   (b) $f^2 = I$,

   (c) $rfrf = I$,

   imply that there are precisely six equivalence classes of words. The way to demonstrate this is to prove that an arbitrary word can be rewritten in the form $r^a f^b$ where r is 0, 1, or 2 and b is 0 or 1. Using the graph for D_3 explain the graphical significance of this result (pages 64–67).

5. List the defining relations for the tetrahedral group A_4. Draw the graph for this group and label each vertex with a member of an equivalence class. Exhibit an isomorphism between the vertex labels and the even permutations of the symbols 1, 2, 3, and 4 (pages 118–119, but the last part is not made explicit).

6. Start with a transparency (provided in class) with the graph for A_5. Show that the permutations $r = (12345)$ and $f = (12)(34)$ satisfy the defining relations for this group. Attach labels to several other vertices of the graph, both as products of generators and as permutations. (pages 167–169)

7. Exhibit a set of defining relations and a graph for the symmetry group of the cube. Here is a verbal description of how to draw the graph.

   (a) Draw a large outer square and a small inner square.

   (b) Join each vertex of the outer square to the nearest vertex of the inner square using two dotted edges, between which is inserted an intermediate square with two of its diagonally-opposite vertices connected to the dotted edges.

   (c) You should now have six squares, all with solid edges. For the outer square, arrows on these edges go counterclockwise. For the inner square and intermediate squares, they go clockwise.

   (d) Join up the remaining vertices of the intermediate squares using four dotted edges. The 12 edges that join different squares (corresponding to the order 2 generator) should all be dotted.

The solid edges correspond to a generator r that satisfies the defining relation $r^4 = 1$. The dotted edges correspond to a generator f that satisfies the defining relation $f^2 = 1$. The third defining relation is $(rf)^3 = 1$.

Using groups.exe, it is easy to associate a permutation with each vertex of this graph. Choose, for example $r = (1234)$ and $f = (12)$. Choose a vertex, in the top left corner, say, to correspond to the identity element. Then just keep multiplying by r and f to reach all the other vertices.