MATHEMATICS S-152, SUMMER 2005
THE MATHEMATICS OF SYMMETRY

Outline #2 (Permutations)

Reading. Biggs, Sections 5.2-5.4, 10.6, 12.5, 12.6, and 21.1. The topic is continued in the rest of chapter 21, but that must wait until you know some group theory.

To gain more experience with permutations, play around with *groups.exe*, a Windows application program that you can download from the course Web site. It is helpful to have at hand your models of the Platonic solids and some regular polygons.

1. Define "injection," "surjection," and "bijection," and illustrate each concept with a diagram for the case of a function a from a finite set X to a finite set Y. Then define "permutation" and show how to represent as a function the permutation of the set $\{1, 2, 3, 4, 5, 6\}$ that is written in cycle notation as (136)(25). Show that there are $n!$ distinct permutations of a set of n elements. (Biggs, Sections 5.2 and 10.6)

2. From the definition of a permutation as a bijection, show that the composition of two permutations is a permutation and that every permutation has an inverse. (Biggs, Sections 5.3, 5.4, and 10.6, though part is relegated to the exercises)

3. Show that there is a one-to-one correspondence between all permutations of the set $\{1, 2, 3, 4\}$ and the set of rotational symmetries of the octahedron. Identify each the symbols 1, 2, 3, 4 with a pair of opposite faces of the octahedron. You can choose how to associate numbers with colors. Using the alias approach and axes fixed in the octaahedron, carry out rotations that illustrate the result

$$(124)(134) = (13)(24).$$

(groups.exe and experiment)

4. Start with an icosahedron whose 30 edges are colored with five distinct colors as specified by your "icosahedron kit." Take red = 1, yellow = 2, green =3 , blue = 4, fifth color = 5. Using the alias approach (e.g. you rotate the icosahedron and every location where there used to be a red edge is now occupied by a yellow edge, so f(red) = yellow or $f(1) = 2$) determine the permutation that corresponds to each of the following.

   (a) A rotation about an axis through a pair of vertices.

   (b) A rotation about an axis through the center of a pair of opposite faces.

   (c) A rotation about an axis through the midpoints of two opposite edges.

   (groups.exe and experiment)

5. In outline 1, we counted the number of permutations of the five symbols 1,2,3,4,5 with each of the following cycle structures:

   (a) like (123)

   (b) like (12345)

   (c) like (12)(34)

   We also counted the number of rotational symmetries for various types of axes of the regular icosahedron. Review these results, and show that there is a one-to-one correspondence between these permutations (plus the identity) and the rotations of the regular icosahedron. (Biggs, sections 12.5 and 12.6)

6. Describe a mindless mechanical way of expressing any permutation as a product of transpositions; e.g., $(12345) = (15)(14)(13)(12)$. Express the "even" permutation (123) as a product of two transpositions and as a product of four transpositions. Then express each of the "odd" permutations (1234) and (123)(45) as a product of three transpositions and as a product of five transpositions. Propose (without proof) a rule to decide whether the number of transpositions will be odd or even. (Biggs, Section 12.6)

7. Prove that when an arbitrary permutation π is followed by a transposition τ, the product $\tau\pi$ must have either one more cycle or one fewer

cycle than π. This is the start of the proof of Theorem 12.6.1 in Biggs. For this proof, include cycles that have only a single element.

8. Complete the proof of Theorem 12.6.1 in Biggs and explain what is meant by an "even" permutation. Show that the rotations of the tetrahedron and of the icosahedron lead only to even permutations, not to odd ones.