MATHEMATICS S-152, SUMMER 2005
THE MATHEMATICS OF SYMMETRY

Outline #3 (Groups)

Reading. The first several sections in Biggs' Chapter 20.

This class introduces us to the formal mathematical structure of a group.

1. Define a *group* as a set with a binary operation on that set satisfying four axioms. List the four axioms. Show that the integers $\mathbb{Z}$ and the rational numbers $\mathbb{Q}$ are groups under addition. Explain why the integers do not form a group under multiplication, while the rational numbers form a group under multiplication only if 0 is deleted from the set.) (Section 20.1)

2. Invent examples of sets with binary operations that are not groups, illustrating which particular axioms fail. Try to find one example for each axiom. Some possibilities:

   - Use subtraction as the operation.
   - Consider the set of non-negative integers under addition.
   - Consider the set of odd integers (both positive and negative) plus zero under addition.
   - Consider the set of odd permutations of the symbols 1,2,3.

3. Show that the collection of permutations on a fixed set of n elements, denoted S_n, is a group when the operation in question is the composition of functions. Define the *cardinality* of a group G to be the number of elements in the group, denoted $|G|$, and show that $|S_n| = n!$. (Sections 10.6 and 21.1.)

4. Classify the symmetries of the equilateral triangle by type, and construct a table of compositions of these symmetries, showing that in fact they form a group under composition. This group, when regarded as a geometric symmetry group, is denoted D_3. Define an *abelian* group, and show that D_3 is not abelian. Use the alibi approach and axes fixed in space, and adopt the names that Biggs uses for the symmetries.

In writing out the complete table of compositions, you can save a lot of time by taking advantage of the fact that each result occurs precisely once in each row and in each column and that the table breaks down into four 3×3 blocks, each involving only "rotations" (through 0, 120, or 240 degrees) or "flips" (about axes in the plane ofthe triangle). (Sections 20.2 and 20.3.)

5. Considering groups defined abstractly, prove that the identity must be unique. Prove that any element in a group has a unique inverse. Prove the cancellation laws. (Section 20.3.)

6. Suppose that we assume only the existence of a left inverse for each element: for any a, there is a b such that $ba = e$, and that we further assume only that e is a left identity, that $ea = a$ for every a. Prove that b is also a right inverse for a. *Hint*: there is some left inverse for b. Call it c and consider $cbab$ to show that $ab = e$) Prove that e is also a right identity: that $ae = e$ for all a (Hint: let b be the inverse (now known to be both the left and right inverse) of a and consider the expression bab.