MATHEMATICS S-152, SUMMER 2005
THE MATHEMATICS OF SYMMETRY

Outline #5 (Subgroups, Cosets, and Quotient Groups)

Reading. Biggs, Chapter 20.

We consider the structure of groups, with special attention to subgroups, generators, and cyclic groups, then explore normal subgroups and quotient groups.

Note. Topics 1 though 6 will be covered on the first quiz. The remaining topics will be covered on the second quiz.

1. Define a *subgroup* of a group as a subset of the given group satisfying the usual group axioms. Show that the closure and inverse axioms imply the identity axiom, and argue that the associativity axiom is automatically implied. Give examples of subgroups of $(\mathbb{Z}, +)$, $(\mathbb{Z}_n, +)$, and S_n. (Section 20.7.)

2. Define the *order* of an element $g \in G$ as the smallest positive integer n such that $g^n = e$, provided that such an n exists. (We say that g has *infinite order* otherwise.) Give examples of the order of various elements in S_5, D_3, and $\mathbb{Z}_7^\times$ under multiplication. (Section 20.4.)

3. Let G be a group, and let $g \in G$ be a fixed element. Consider the set $\langle g \rangle = \{\ldots, g^{-1}, e, g, g^2, g^3, \ldots\}$. Show that $\langle g \rangle$ is a subgroup of G, called the subgroup *generated* by g. Define a *cyclic* group as one generated by a single element in this manner, that is, H is cyclic if there is some $h \in H$ such that $H = \langle h \rangle$. Show that a cyclic group is abelian. Give examples of cyclic subgroups of S_5, D_n, and $\mathbb{Z}_{10}$ under addition. (Section 20.6 and 20.7.)

4. Define the left coset of a subgroup H of group G with respect to an element g of G. For the case where G is the symmetry group of the triangle and H is one of the 180-degree "flips," describe the three cosets of H. Present the same analysis, thinking of the group as the permutations of $\{1, 2, 3\}$ and H as the subgroup generated by the transposition $(1\ 2)$.

5. Using groups.exe, display the left cosets of a couple of subgroups of the symmetry group of the regular icosahedron (Section 20.8). Show that in every case, the left cosets break up the elements of the group into disjoint subsets of equal size. Show that the right cosets also do the same thing, but that in general the left cosets and right cosets are not the same,

6. Prove Theorem 20.8.1 in Biggs, which states that if two left cosets have a single element in common, then they are identical. Show that from this follows Lagrange's Theorem, which states that the order of a subgroup divides the order of the group (Section 20.8).

7. *Conjugation.* If g and a are elements of a group, then $h = aga^{-1}$ is "the conjugate of g with respect to a." Prove the following important properties of conjugation:

   (a) Any power of h is the conjugate of the corresponding power of g.

   (b) Conjugate elements have the same order.

8. Demonstrate conjugacy using groups.exe as follows: with a subgroup displayed, click "Mark Conjugator". The element turns blue and its inverse (if different) turns yellow. Click "Show Conjugate Subgroup" to highlight the conjugate subgroup. Point out illustrations of the following:

   (a) Conjugate permutations have the same cycle structure.

   (b) The conjugates of the elements of a subgroup form a "conjugate subgroup" of the same order.

   (c) For rotation groups of polygons and polyhedra, conjugate subgroups have "equivalent" axes of rotation.

9. Prove in general the following properties of conjugation, which were just demonstrated in specific cases.

   (a) Conjugate permutations have the same cycle structure.

   (b) The conjugates of the elements of a subgroup form a "conjugate subgroup" of the same order.

   (c) For rotation groups of polygons and polyhedra, conjugate subgroups have "equivalent" axes of rotation.

10. A subgroup that is its own conjugate under any element of the group is called a "normal subgroup" or a "self-conjugate subgroup." Show that a subgroup H is normal if and only if the left coset aH equals the right coset Ha for every element a of the group G.

11. Show that for the tetrahedron and the cube, the symmetry group has a normal subgroup but that for the icosahedron it does not. (The easiest way to do this is with groups.exe.) Demonstrate the normal subgroups that you find, both as rotations of the Platonic solid and as permutations.

12. Suppose that H is a normal subgroup of G. Then we can define a group operation on left cosets as follows

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    - For coset A choose any group element a such that $aH = A$.
    - For coset B choose any group element b such that $bH = B$.
    - Define the product AB to be the coset $(ab)H$.

    (a) Prove that this product is well defined, since the coset AB is independent of the specific choice of the elements a and b.

    (b) Verify that all the group axioms are satisfied for this operation.

    So the cosets form a group denoted G/H: the "quotient group." Notice that $|G| = |H||G/H|$

13. Let G be the group of the integers under addition, and let H be the subgroup of integers that are divisible by 3. Show that the three cosets are just the equivalence classes [0], [1], and [2] and that operation of forming the "product" of two of these cosets, as just defined, is the same as the operation of addition in $\mathbb{Z}_3$.

14. Present two important special cases of normal subgroups and quotient groups, as follows:

    (a) If $|G| = 2|H|$ there are only 2 cosets, and H must be normal. The quotient group has only two elements. Show that this case occurs when $G = S_n$, the group of all permutations of n symbols, and H is the "alternating subgroup" A_n of even permutations.

(b) For the symmetry group of the tetrahedron, show that there is a normal subgroup of order 4, and find the associated quotient group. Choose one of the cosets that does not include the identity, and show that all of its four elements are rotations of a face of the tetrahedron with the same “ handedness” when viewed from the fourth vertex.