MATHEMATICS S-152, SUMMER 2005
THE MATHEMATICS OF SYMMETRY

Outline #7 (Finite Affine Geometry)

References

- Bennett, *Affine and Projective Geometry*, Chapter 3. This book, available in Cabot Library, covers all the proofs and has nice diagrams.
- "Faculty Senate Affine Geometry" (attached). This has all the steps for each proof, but no diagrams. There are numerous references to diagrams on the course web site; however, and the combination of this document and the web site should be all that you need.
- The course web site, AffineDiagrams folder. This has links that bring up step-by-step diagrams for all key results. These diagrams will be available in class, and you are welcome to use them instead of drawing new diagrams on the blackboard.
- The Windows application program, *affine.exe*, which can be downloaded from the course web site.
- Data files for the small, medium, and large affine senates. These accompany affine.exe, since they are data files read by that program. The file *affine.zip* has everything.
- PHP version of the affine geometry software. This program, written by Harvard undergraduate Luke Gustafson, runs directly off the web and generates nice diagrams whenever you use affine geometry to do arithmetic. It also has nice built-in documentation. Choose the PHP-Programs folder on the web site.

Finite Affine Geometry

1. State the first four of the five axioms for a finite affine plane, using the terms "instructor" and "committee" instead of "point" and "line." For A4 (Desargues), draw diagrams (or show the ones on the web site) to

illustrate the two cases (three parallel lines and three concurrent lines) in the Euclidean plane. For the first of these two cases, outline a proof in Euclidean geometry (for which this is a theorem, not an axiom.)

2. For the second case of A4 (Desargues), give a proof using vectors. Here is an outline of the proof:

   (a) Draw vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ from point O (where AA', BB', and CC' intersect) to A, B and C, respectively.

   (b) Call the vector from O to A' $\alpha\vec{u}$.

   (c) Use the fact that $A'B'$ is parallel to AB to show that the vector from O to B' is $\alpha\vec{v}$.

   (d) Use the fact that $A'C'$ is parallel to AC to show that the vector from O to C' is $\alpha\vec{w}$.

   (e) Use vector subtraction to calculate the vectors BC and $B'C'$ and show that they are parallel.

3. State the fifth axiom A5 (Pappus) for a finite affine plane, and draw a diagram to illustrate it in the Euclidean case. Give a proof using vectors in Euclidean geometry (for which this is again a theorem, not an axiom). Here is an outline of the proof.

   (a) Draw linearly independent vectors $\vec{v}$ and $\vec{w}$ from a point O to A and D respectively.

   (b) Construct point B at $O + \alpha\vec{v}$ and point E at $O + \beta\vec{w}$

   (c) Construct C on line OAB with EC parallel to DB.

   (d) Construct F on line ODE with BF parallel to AE.

   (e) Show that AD and CF are parallel vectors.

4. Draw a diagram to show the smallest possible system satisfying these axioms, the "tiny affine senate" with four instructors Al, Bo, Cy, and Di and six committees. Point out how the Axioms A1–A3 are satisfied, and point out an unavoidable misleading feature of your Euclidean diagram, namely, that parallel committees are not always represented by parallel lines.

5. Show how to use "parallel committees" to establish a bijection between any two (parallel or intersecting) committees, and thereby prove that all committees have the same number of members n.

6. Show that each instructor serves on $n + 1$ committees, and conclude that there are $n(n + 1)$ committees and n^2 instructors.

7. Using the data for the large affine faculty senate (in the notes, and also built into the software on the web site), construct an example of the second case of Axiom A4 where every instructor and every committee in the diagram has a name. The program affine.exe will help you to do this.

8. Using the data for the large affine faculty senate (in the notes, and also built into the software on the web site), construct an example of Axiom A5 where every instructor and every committee in the diagram has a name. The secret of doing this is to choose two intersecting committees, select two instructors (but do not use the point of intersection) on each of them, and then use the parallelism assumptions in the axiom to identify the correct third instructor on each committee. The diagram on the web site shows the right sequence of steps.

9. Draw a diagram to illustrate the algorithm for adding two members of a committee. You can illustrate the steps by using the diagram on the web site. Prove that the answer is independent of the choice of auxiliary instructor. Do this carefully for the special case where the two alternative auxiliary instructors B and B' serve on a committee that includes O, then show a diagram on the web site to indicate what one extra step is needed in the general case.

10. Show that the addition of two members of a committee is associative. There is a useful diagram on the web site.

11. Show that for the addition of two members of a committee

    (a) Instructor O is a right identity: $C + O = C$.

    (b) Every instructor A has a right inverse V with $A + V = O$.

    There are useful diagrams on the web site.

Conclude that the instructors on a committee form a group under addition.

12. Prove the converse of Axiom A4 (Theorem 11).

13. Use the theorem that was just proved to show that addition of two members of a committee is commutative. There is a useful diagram on the web site.

14. Draw a diagram (or show one from the web site) to illustrate the algorithm for multiplying two members of a committee. You can illustrate the steps by using the program affine.exe. Prove that the answer is independent of the choice of auxiliary instructor. It is okay just to do this for the special case where the two alternative auxiliary instructors B and B' serve on a committee that includes I, since the proof of the general case is very similar.

15. Show that the multiplication of two members of a committee satisfies the last three group axioms.

    (a) It is associative (but don't take the time to prove this, since the proof is almost identical to the proof for addition).

    (b) Instructor I is a right identity: $CI = C$.

    (c) Every instructor A except O has a right inverse V with $AV = I$.

16. Use Axiom A5 to prove that multiplication is commutative. There is a useful diagram on the web site.

    (The proof of the distributive law, all that is needed to complete the list of axioms for a field, is omitted because it involves a very complicated diagram but no new ideas).

17. Using affine.exe or the PHP version on the web site, choose any committee of the small affine senate and generate the addition and multiplication tables for it. Show how to associate the committee members with congruence classes [0], [1], and [2] in such a way that it is clear that the committee is the same field as $\mathbb{Z}_3$.

18. Use affine.exe or the PHP version on the web site to generate the addition and multiplication tables for a committee of the medium affine

faculty senate, and show two different ways of identifying the members of the committee with the four elements $[0]$, $[1]$, $[x]$, and $[x+1]$ of the finite field $\mathbb{F}_4$. (The fact that there are two equally good ways to do this will become significant in a couple of weeks.)

The last three topics are exercises for the entire class. To emphasize the point that the results are valid for any choice of origin and axes, several arbitrary choices have not been specified in advance.

19. Identify the blackboard with $\mathbb{R}^2$. The blackboard is to be regarded as an affine plane, not a Euclidean plane, so pay no attention to angles, and do not compare the lengths of segments on lines that are not parallel.

    (a) Choose an origin O.

    (b) Choose a line through O as the x-axis, and specify the multiplicative identity $(1, 0)$ on this line.

    (c) Choose another line through O as the y-axis, and specify the multiplicative identity $(0, 1)$ on this line. To avoid being influenced by concepts that are Euclidean rather than just affine, it is a good idea draw the y-axis so that it is not perpendicular to the x-axis and to choose the multiplicative identity for y at a different distance from O than the multiplicative identity for x.

    (d) Locate the point whose coordinates are $(1, 1)$ and the line that passes through $(0, 0)$ and $(1, 1)$.

    (e) Locate the point $(2, 2)$ and show how to use parallel lines to locate the points $(2, 0)$ and $(0, 2)$.

    (f) Choose an arbitrary point and show how to use parallel lines to associate coordinates (x, y) with it.

    (g) Consider the line that passes through $(1, 0)$ and $(0, 2)$. Show how to determine all the points on this line, both as a point plus an arbitrary multiple of a vector and as a linear equation satisfied by the coordinates of any point (x, y) on the line.

20. Associating a pair of coordinates from $\mathbb{Z}_3$ with every instructor in the small affine senate.

    (a) Choose any instructor to be the origin O.

(b) Choose any committee through O as the x-axis, and specify the multiplicative identity $([1], [0])$ on this committee.

(c) Choose any other committee through O as the y-axis, and specify the multiplicative identity $([0], [1])$ on this committee.

(d) Identify the instructor whose coordinates are $([1], [1])$ and the committee that passes through $([0], [0])$ and $([1], [1])$.

(e) Identify the instructor $([2], [2])$ and show how to use parallel committees to locate the instructors $([2], [0])$ and $([0], [2])$.

(f) Choose an instructor whose coordinates are not yet determined, and show how to use parallel committees to associate coordinates (x, y) with this instructor.

(g) Draw a diagram showing the nine instructors arranged in a square pattern with the x-axis horizontal and the y-axis vertical, showing the coordinates for each instructor.

(h) Consider the committee that includes $([1], [0])$ and $([0], [2])$. Show how to determine all the instructors on this committee, both as a point plus an arbitrary multiple of a vector and as a linear equation satisfied by the coordinates of any point (x, y) on the line.

(i) Determine which committee satisfies the equation $x + [2]y = [2]$.

21. Associating a pair of coordinates from $\mathbb{F}_4$ with every instructor in the medium affine senate. To avoid reusing the symbols x and y, call the elements of this field $[0]$, $[1]$, $[u]$, and $[u + 1]$.

    (a) Choose any instructor to be the origin O.

    (b) Choose any line through O as the x-axis, and specify the multiplicative identity $([1], [0])$ on this line.

    (c) Choose any other line through O as the y-axis, and specify the multiplicative identity $([0], [1])$ on this line.

    (d) Identify the instructor whose coordinates are $([1], [1])$ and the committee that passes through $([0], [0])$ and $([1], [1])$.

    (e) Choose the instructor $([u], [u])$ (this really does involve an arbitrary choice) and show how to use parallel committees to locate the points $([u], [0])$ and $([0], [u])$.

(f) Choose an instructor whose coordinates are not yet determined, and show how to use parallel committees to associate coordinates (x, y) with this instructor.

(g) Draw a diagram showing the sixteen instructors arranged in a square pattern with the x-axis horizontal and the y-axis vertical, showing the coordinates for each instructor.

(h) Consider the committee that includes $([1], [0])$ and $([0], [u])$. Show how to determine all the instructors on this committee, both as a point plus an arbitrary multiple of a vector and as a linear equation satisfied by the coordinates of any point (x, y) on the line.

(i) Determine which committee satisfies the equation $[u]x + [u+1]y = [u]$.