MATHEMATICS S-152, SUMMER 2005
THE MATHEMATICS OF SYMMETRY

Outline #9 (Group Isomorphisms)

References

We look at 2×2 matrices over finite fields and discover groups of these matrices that are isomorphic to the symmetry group of the icosahedron and to permutation groups. No textbook covers this topic. A set of notes is attached.

1. Using a model of the regular icosahedron or of the regular dodecahedron, demonstrate that every rotation permutes the five colors of the edges, and that

   - an element of order 5 corresponds to a permutation like (12345) that leaves no color fixed.
   - an element of order 2 corresponds to a permutation like (12)(34) that leaves one color fixed.
   - an element of order 3 corresponds to a permutation like (123) that leaves two colors fixed.

   Use the alias interpretation.

2. Consider the vector space $\mathbb{F}_4^2$. List the three vectors in each of the five one-dimensional subspaces, and explain how these can be identified with the "committees through the origin" in the medium affine faculty senate. For the sake of concreteness, lay out the senate like this:

education	football	gifts	housing	
Mike$(0, x+1)$	Neil$(1, x+1)$	Owen$(x, x+1)$	Phil$(x+1, x+1)$	diversity
Irma$(0, x)$	Jane$(1, x)$	Kate(x, x)	Lynn$(x+1, x)$	compensation
Emma$(0, 1)$	Fred$(1, 1)$	Greg$(x, 1)$	Herb$(x+1, 1)$	budget
Adam$(0, 0)$	Beth$(1, 0)$	Carl$(x, 0)$	Dave$(x+1, 0)$	athletics

3. Prove that for a 2×2 matrix A , the eigenvalues are determined by the determinant, $\det A$, and the trace, $\text{tr}A$. Then, by considering the four different ways that a 2×2 matrix over $\mathbb{F}_4$ can have determinant 1, count how many matrices in $SL(2, \mathbb{F}_4)$ have trace 0, trace 1, trace x, and trace $x+1$.

4. By constructing and solving the "characteristic equation,"

$$\det(A - \lambda I) = 0,$$

show that

   (a) a matrix in $SL(2, \mathbb{F}_4)$ with trace 0 has a single eigenvalue and therefore would have to correspond to an element of A_5 like (12)(34) with order 2 or be the identity.

   (b) a matrix in $SL(2, \mathbb{F}_4)$ with trace 1 has two eigenvalues and therefore would have to correspond to an element of A_5 like (123) with order 3.

   (c) a matrix in $SL(2, \mathbb{F}_4)$ with trace x or $x + 1$ has no eigenvalues and therefore would have to correspond to an element of A_5 like (12345) with order 5.

5. With the aid of the program SL2F.exe (on the Web site), show examples of the action of matrices with order 2, order 3, and order 5 on the one-dimensional subspaces of $\mathbb{F}_4^2$. Interpret each of these as a "collineation" of the medium affine senate that carries instructors into instructors and committees into committees while holding the "origin" Adam fixed.

6. Suppose that G is a group with operation $\times$ and H is a group with operation $*$. Let f be a bijection from G to H that satisfies $f(g \times g') = f(g) * f(g')$ for any pair of elements g and g' in G. The bijection f is called an *isomorphism* between G and H. Prove the following results.

   (a) The inverse of f, $f^{-1} : H \longrightarrow G$, is also an isomorphism.

   (b) f maps the identity of G to the identity of H.

   (c) f maps inverses into inverses: $f(g^{-1}) = f(g)^{-1}$.

   (d) If f is an isomorphism from G to H and j is an isomorphism from H to K, then the composition $q = j \circ f$ is an isomorphism from G to K.

7. Using the definition of isomorphism that was just presented, show that there is an isomorphism between the rotation group of the dodecahedron and A_5, an isomorphism between $SL(2, \mathbb{F}_4)$ and A_5, and therefore an isomorphism between the rotation group of the dodecahedron and $SL(2, \mathbb{F}_4)$. All that you need to do is to describe the "bijection f" from the previous topic.

8. Using a diagram of the dodecahedron (in the notes, and available on a transparency), exhibit explicitly the isomorphism between the rotation group of the dodecahedron and $SL(2, \mathbb{F}_4)$. Show two or three examples of how you can associate a matrix with a rotation and a vector with an edge in such a way that the action of the matrix on the vector correctly predicts the effect of the rotation on the edge.

9. Go back to the coordinatization of the medium affine faculty senate

education	football	gifts	housing	
Mike$(0, x+1)$	Neil$(1, x+1)$	Owen$(x, x+1)$	Phil$(x+1, x+1)$	diversity
Irma$(0, x)$	Jane$(1, x)$	Kate(x, x)	Lynn$(x+1, x)$	compensation
Emma$(0, 1)$	Fred$(1, 1)$	Greg$(x, 1)$	Herb$(x+1, 1)$	budget
Adam$(0, 0)$	Beth$(1, 0)$	Carl$(x, 0)$	Dave$(x+1, 0)$	athletics

Demonstrate by example that the bijection that results from interchanging the top two rows and the right two columns (i.e. interchanging x and $x+1$) is a collineation in the sense that it maps committees into committees. By looking at the five committees of which Adam is a member, show that it corresponds to a permutation like (4 5) which is an element of S_5 but not of A_5.

10. By considering a dodecahedron with six colors used to paint the six pairs of opposite faces (or an icosahedron with six colors or numbers attached to pairs of opposite vertices), show that the rotation group of the dodecahedron (and hence also A_5) is isomorphic to a subgroup of A_6. With the aid of groups.exe, using the tab "Icosahedron 6," give examples of this isomorphism, and indicate what cycle structure corresponds to the elements of order 2, order 3, and order 5 respectively.

11. Enumerate the one-dimensional subspaces of $\mathbb{Z}_5^2$, naming the elements 0, 1, 2, -2, and -1. By considering the five different ways that a 2×2 matrix over $\mathbb{Z}_5$ can have determinant 1, count how many matrices in $SL(2, \mathbb{Z}_5)$ have trace 0, trace 1, trace 2, trace -2, and trace -1. Give a convincing argument that although $SL(2, \mathbb{Z}_5)$ has the same number of elements as S_5, the two groups are not isomorphic. Hint: look at

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

12. By constructing and solving the "characteristic equation,"

$$\det(A - \lambda I) = 0,$$

show that

- a matrix in $SL(2,\mathbb{Z}_5)$ with trace 0 has two eigenvalues and therefore would have to correspond to an element of A_6 like (12)(34) with order 2.
- a matrix in $SL(2,\mathbb{Z}_5)$ with trace $+1$ or -1 has no eigenvalues and therefore would have to correspond to an element of A_6 like (123)(456) with order 3.
- a matrix in $SL(2,\mathbb{Z}_5)$ with trace $+2$ or -2 has one eigenvalue and therefore would have to correspond to an element of A_6 like (12345) with order 5.

13. Show that the subgroup Z of $SL(2,\mathbb{Z}_5)$ consisting of the identity I and the matrix $-I$ is a normal subgroup. Describe the cosets of Z, show that there are 60 of them, and state a rule for selecting one of the two elements of a coset. Illustrate the action of the quotient group $PSL(2,\mathbb{Z}_5) = SL(2,\mathbb{Z}_5)/Z$ on the one-dimensional subspaces of Z_5^2.

14. The large affine faculty senate can be coordinatized like this:

insurance	junior faculty	football	gifts	housing	
Nancy(-2,2)	Oscar(-1,2)	Kevin(0,2)	Larry(1,2)	Maria(2,2)	compensation
Isaac(-2,1)	James(-1,1)	Frank(0,1)	Gavin(1,1)	Helen(2,1)	budget
Danny(-2,0)	Emily(-1,0)	Alice(0,0)	Betty(1,0)	Chris(2,0)	athletics
Xenia(-2,-1)	Yoric(-1,-1)	Uriah(0,-1)	Viola(1,-1)	Wally(2,-1)	education
Sally(-2,-2)	Terry(-1,-2)	Patty(0,-2)	Quint(1,-2)	Ralph(-2,-2)	diversity

Prove that any matrix in $GL(2,\mathbb{Z}_5)$ (which maps instructors into instructors by acting on their coordinates) also maps committees into committees, but that the subgroup Z that consists of non-zero multiples of the identity maps each committee into itself. (Hint: each committee satisfies a linear equation). Show that the quotient group of $PGL(2,\mathbb{Z}_5) = GL(2,\mathbb{Z}_5)/Z$ has the same number of elements as S_5, that it includes odd as well as even permutations of the six committees to which Alice belongs, and that the cosets containing matrices with determinant $+1$ and -1 correspond to even permutations while the cosets containing matrices with determinant $+2$ and -2 correspond to odd permutations.