Problem Set 6

March 29, 2016

Let $(\Omega, \mathcal{F}, P)$ be a probability space.

(1) Let $X : \Omega \to \mathbf{R}$ be a random variable. For $n \geq 0$, we say that X has *finite nth moment* if $\mathbf{E}[|X|^n] < \infty$. Show that if X has finite nth moment, then it also has finite mth moment for $m \leq n$.

(2) Let $X : \Omega \to \mathbf{R}_{\geq 0}$ be a random variable with finite 2nd moment. The *variance* of X is defined by $\text{Var}(X) = \mathbf{E}[X^2] - \mathbf{E}[X]^2$. Show that if $\text{Var}(X) = 0$, then X is constant almost everywhere: that is, there exists a real number t for which the event $\{\omega \in \Omega : X(\omega) = t\}$ has probability 1.

For $n \geq 0$, we say that a subset $A \subseteq \mathbf{R}^n$ is a *Borel set* if it belongs to the σ-algebra generated by sets of the form $(a_1, b_1) \times \cdots \times (a_n, b_n)$.

(3) Show that a function $X : \Omega \to \mathbf{R}$ is a random variable if and only if, for each Borel set $A \subseteq \mathbf{R}$, the set $\{\omega \in \Omega : X(\omega) \in A\}$ belongs to $\mathcal{F}$.

(4) Let $f : \mathbf{R}^n \to \mathbf{R}$ be a continuous function, and suppose that $X_1, \ldots, X_n : \Omega \to \mathbf{R}$ are random variables. Show that the function

$$X : \Omega \to \mathbf{R} \qquad X(\omega) = f(X_1(\omega), X_2(\omega), \ldots, X_n(\omega))$$

is also a random variable (hint: show that if $A \subseteq \mathbf{R}$ is a Borel set, then $f^{-1}(A) \subseteq \mathbf{R}^n$ is also a Borel set).

(5) Suppose that $\mathcal{F}$ is the σ-algebra generated by a semi-field $\mathcal{F}_0 \subseteq \mathcal{F}$. For each subset $A \subseteq \Omega$, let $\mu^*(A)$ denote the the outer measure of A with respect to $\mathcal{F}_0$ (as in Lecture 22). Recall that A is said to be *measurable* if $\mu^*(X) = \mu^*(X \cap A) + \mu^*(X \cap A^c)$ for every subset $X \subseteq \Omega$. Prove that A is measurable if and only if there exist events $A_-, A_+ \in \mathcal{F}$ satisfying

$$A_- \subseteq A \subseteq A_+ \qquad P(A_-) = P(A_+).$$