★ On homework #26: Can use fact that $\omega(s_\lambda) = s_{\lambda'}$.

155 Lecture 15

Today: Murnaghan Nakayama Rule, a combinatorial formula for the value of irreds. characters.

__Lemma:__ For any $f \in \Lambda$, $\langle f s_\nu, s_\lambda \rangle = \langle f, s_{\lambda/\nu} \rangle$.
In particular, $\langle s_\mu s_\nu, s_\lambda \rangle = \langle s_\mu, s_{\lambda/\nu} \rangle$.

__Proof:__ Use classical def. of Schur functions, that $s_\lambda(x_1, \dots, x_n) = \dfrac{a_{\lambda+\delta}}{a_\delta}$ where

$$a_\alpha = \det\left(x_i^{\alpha_j}\right)_{i,j=1}^{n}, \text{ and } \delta = (n-1, n-2, \dots, 0).$$

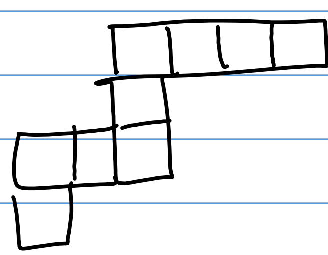
Proof is tedious so omit: but see EC2.

__Lemma:__ $\langle p_\lambda, p_\mu \rangle = z_\lambda^{-1} \delta_{\lambda\mu}$.

[Straightforward; use lemma saying $\{u_\lambda\}$ & $\{v_\lambda\}$ are dual bases iff $\sum_\lambda u_\lambda(x) v_\lambda(y) = \prod_{i,j} (1 - x_i y_j)^{-1}$

__Def:__ A __rim hook__ or __border strip__ is a skew diagram which is edgewise connected & contains no 2×2 subset of cells 

__Ex:__



Define the <u>height</u> $ht(B)$ of a border strip to be one less than its number of rows.

<u>Theorem:</u> For any $\mu \in \mathrm{Par}$ and $r \in \mathbb{N}$,

$$s_\mu p_r = \sum_{\lambda} (-1)^{ht(\lambda/\mu)} s_\lambda,$$

summed over all partitions $\lambda \supseteq \mu$ for which λ/μ is a border strip of size r.

As usual if $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ write $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. If $w \in S_n$ we use $w(x^\alpha)$ to denote $x_1^{\alpha_{w(1)}} \cdots x_n^{\alpha_{w(n)}}$.

<u>Proof:</u> Let $\delta = (n-1, n-2, \dots, 0)$ & let all functions be in variables $x_1, x_2, \dots, x_n$.

$$a_\alpha = \det\left(x_i^{\alpha_j}\right)_{i,j=1}^{n} = \sum_{w \in S_n} \varepsilon_w \, w(x^\alpha) \qquad (*)$$

where $\varepsilon_w = \begin{cases} 1 & \text{if } w \text{ an even perm.} \\ -1 & \text{if } w \text{ an odd perm.} \end{cases}$

Let $\alpha = \mu + \delta$ and multiply $(*)$ by $p_r = \sum_{i=1}^{n} x_i^r$

On LHS get $a_{\mu+\delta} p_r$. On RHS get

$$\sum_{i=1}^{n} x_i^r \sum_{w\in S_n} \varepsilon_w \, w(x^{\mu+\delta}) =$$

$$\sum_{i=1}^{n} \sum_{w\in S_n} \varepsilon_w \, x_i^r \, w(x^{\mu+\delta})$$

If we let ε_i be the sequence w/ a 1 in the i^{th} place and 0 elsewhere, this is equal to

$$\sum_{i=1}^{n} \sum_{w\in S_n} \varepsilon_w \, w(x^{\mu+\delta+r\varepsilon_i}) =$$

$$\sum_{i=1}^{n} a_{\mu+\delta+r\varepsilon_i}. \qquad \text{So } a_{\mu+\delta}\, p_r = \sum_{i=1}^{n} a_{\mu+\delta+r\varepsilon_i}$$

Consider the sequence $\mu+\delta+r\varepsilon_q$.
$\mu+\delta$ is a partition w/ distinct parts — $(\mu_1+n-1, \mu_2+n-2, \dots, \mu_{n-1}+1, \mu_n)$ —

but $\mu+\delta+r\varepsilon_q$ might not be: parts might not be decreasing & 2 could be equal.

Arrange its parts in decreasing order. If two parts are equal, $a_{\mu+\delta+r\varepsilon_q} = 0$ (because 2 rows in the determinant will be equal)

Otherwise, there is some $p \le q$ s.t.

$$\mu_{p-1} + n-(p-1) > \mu_q + n - q + r > \mu_p + n - p$$

$\mu_q + (n-q) + r$

& the decreasing rearrangement of $\mu + \delta + r\varepsilon_q$ is:

$\mu_1 + n-1, \mu_2 + n-2, \dots, \mu_{p-1} + n-(p-1), \mu_q + n-q+r, \mu_p + n-p, \dots, \mu_{q-1} + n-(q-1), \mu_{q+1} + n-(q+1), \dots, \mu_n$.

Subtracting off δ gives the partition

$\lambda := (\mu_1, \mu_2, \dots, \mu_{p-1}, \mu_q + p-q+r, \mu_p + 1, \dots, \mu_{q-1}+1, \mu_{q+1}, \dots, \mu_n)$.

So in this case, $a_{\mu+\delta+r\varepsilon_i} = (-1)^{q-p} a_{\lambda+\delta}$ ⊛

comes from shifting a part of the partition $q-p$ places.

Claim:
Such partitions λ are exactly those for which λ/μ is a border strip B of size r and $q-p$ is just $ht(B)$.

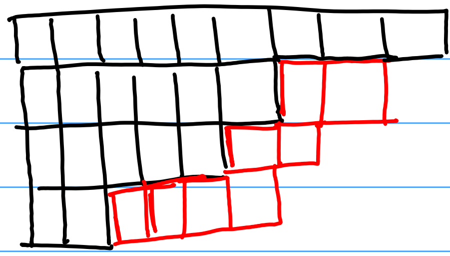
$\lambda/\mu = (0, \dots, 0, \mu_q - \mu_p + p - q + r, \mu_p - \mu_{p+1} + 1, \mu_{p+1} - \mu_{p+2} + 1, \dots, \mu_{q-1} - \mu_q + 1)$

- Total # boxes is: $\mu_q - \mu_p + \mu_p - \mu_{p+1} + \mu_{p+1} - \mu_{p+2} \dots + \mu_{q-1} - \mu_q +$
$+ p - q + r + \underbrace{1+1+\dots+1}_{q-p}$

$= r$

- There are $q-p$ parts in λ/μ so ht is $q-p$.

- To check it's a border strip, need to make sure that whenever $\lambda_i > \mu_i$ & $\lambda_{i+1} > \mu_{i+1}$, the length λ_{i+1} is <u>exactly</u> $\mu_i + 1$.



This description coincides exactly w/ our expr. for λ:

$$\lambda := (\mu_1, \mu_2, \ldots, \mu_{p-1}, \mu_q + p - q + r, \mu_p + 1, \ldots, \mu_{q-1} + 1, \mu_{q+1}, \ldots, \mu_n)$$

We have $a_{\mu+\delta} p_r = \sum_{i=1}^{n} a_{\mu+\delta+r\epsilon_i}$ where either $a_{\mu+\delta+r\epsilon_i} = 0$

OR $a_{\mu+\delta+r\epsilon_i} = (-1)^{q-p} a_{\lambda+\delta}$, where

λ/μ a border strip of size r & height $q-p$.

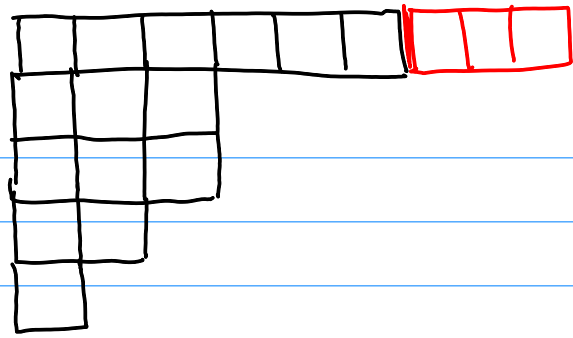
$\therefore$ $a_{\mu+\delta} p_r = \sum_{\lambda} (-1)^{ht(\lambda/\mu)} a_{\lambda+\delta}$. (where)

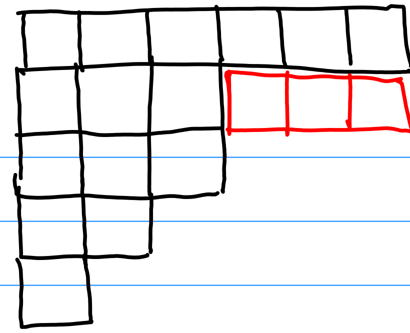
Dividing both sides by a_δ gives

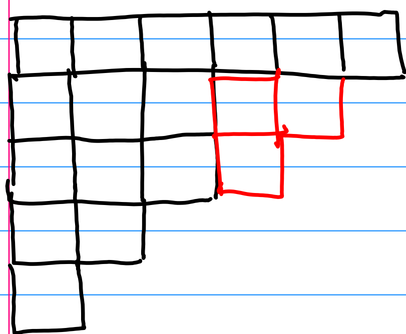
$$s_\mu p_r = \sum_{\lambda} (-1)^{ht(\lambda/\mu)} s_\lambda \quad \left(\text{where } \lambda/\mu \text{ border strip of size } r\right)$$

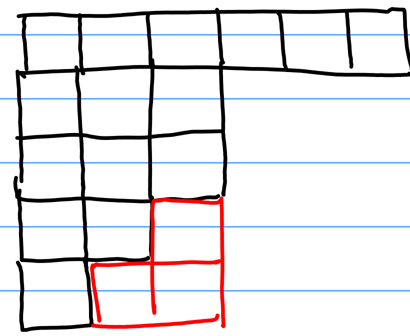
$\blacksquare$

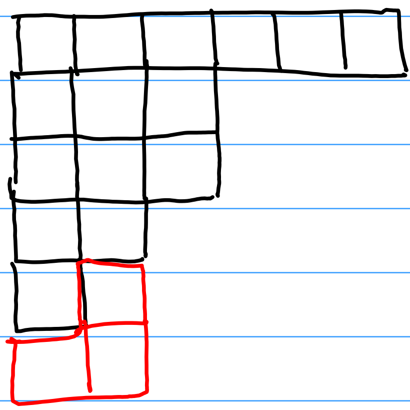
Ex: Say $\mu = 63321$. The border strips of size 3 that can be added to μ are:

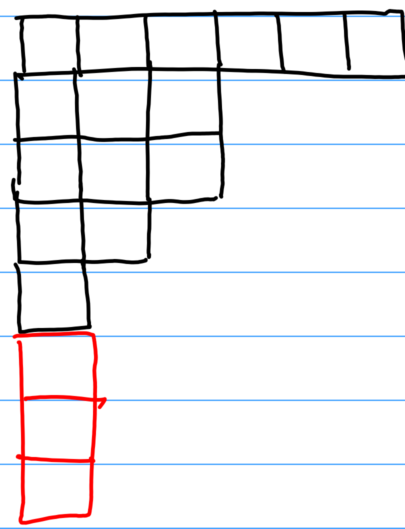












So $s_{63321}\, p_3 = s_{93321} + s_{66321} - s_{65421} - s_{63333} - s_{633222} +$

$s_{63321111}$

Def: Let α be a weak composition of n. Then a **border strip tableau** of shape λ/μ & type α is sequence $\mu = \lambda^0 \subseteq \lambda^1 \subseteq \dots \subseteq \lambda^r \subseteq \lambda$ of partitions s.t. each skew shape λ^i/λ^{i+1} is a border strip of size α_i.

Example:

1	1	1	1	6	6	6
1	2	2	5	6		
3	3	5	5	6		
3	5	5	6	6		

Define the **height** $ht(T)$ to be

$$ht(T) = ht(B_1) + \dots + ht(B_k)$$ where the B_i are the border strips of T.

So in example, height is 7 & type is (5, 2, 3, 0, 5, 7). ASK!

Define $\psi^{\lambda/\mu}(\alpha) = \sum_T (-1)^{ht(T)}$, where sum is over all border-strip tableaux of shape λ/μ and type α.

Theorem: $s_\mu p_\alpha = \sum_\lambda \psi^{\lambda/\mu}(\alpha) s_\lambda$

Proof: We have that

$(*)$ $s_\mu p_r = \sum_\lambda (-1)^{ht(\lambda/\mu)} s_\lambda$ (where λ/μ border strip of size r).

$p_\alpha = p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_k}$, so we repeatedly apply $(*)$. ∎

Let $\mu = \emptyset$. Then Theorem $\Rightarrow$

Cor: $p_\alpha = \sum_\lambda \psi^\lambda(\alpha) s_\lambda$ where

$\psi^\lambda(\alpha) = \sum_T (-1)^{ht(T)}$, where sum is over all border-strip tableaux of shape λ and type α.

Another consequence of Theorem:

Cor: $s_{\lambda/\mu} = \sum_\nu z_\nu^{-1} \psi^{\lambda/\mu}(\nu) p_\nu$.

Proof: Theorem says $s_\mu p_\nu = \sum_\lambda \psi^{\lambda/\mu}(\nu) s_\lambda$

Taking inner product with s_λ, get

$$\psi^{\lambda/\mu}(\nu) = \langle s_\mu p_\nu, s_\lambda \rangle$$
by Lemma
$$= \langle p_\nu, s_{\lambda/\mu} \rangle$$

Now write $s_{\lambda/\mu} = \sum_\nu c_\nu p_\nu$ for coeffs c_ν.

To solve for c_ν, Note that

$$\langle s_{\lambda/\mu}, p_\mu \rangle = \langle \sum_\nu c_\nu p_\nu, p_\mu \rangle = \sum_\nu c_\nu \langle p_\nu, p_\mu \rangle.$$

Recall lemma: $\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}$.

$$\langle s_{\lambda/\mu}, p_\mu \rangle = z_\mu c_\mu \Rightarrow c_\mu = z_\mu^{-1} \langle p_\mu, s_{\lambda/\mu} \rangle.$$

$$\Rightarrow c_\nu = z_\nu^{-1} \psi^{\lambda/\mu}(\nu).$$

This proves Corollary $\blacksquare$

Taking $\mu = \emptyset$, we get:

Cor: $$s_\lambda = \sum_\nu z_\nu^{-1} \psi^\lambda(\nu) \; p_\nu.$$

Recall from previous lecture that

$s_\lambda = \sum_\nu z_\nu^{-1} \chi^\lambda(\nu) p_\mu$, where χ^λ the char of V_λ.

This proves that $\chi^\lambda(\nu) = \psi^\lambda(\nu)$!

So we have proved the following:

<u>Theorem:</u> $\chi^{\lambda}(\alpha) = \sum_{T} (-1)^{ht(T)}$

summed over all border strip tableaux of shape λ/μ + type α. $\blacksquare$

Now: a little about rep theory of $GL(n, \mathbb{C})$

$GL(n, \mathbb{C})$ = general linear gp
= $n \times n$ invertible cx matrices.

If V is an n-dim cx v-space, then after choosing ordered basis for V we can identify $GL(n, \mathbb{C})$ w/ the group $GL(V)$ of invertible linear transformations $A: V \to V$

<u>Def:</u> A <u>linear representation</u> of $GL(V)$ is a homomorphism $\varphi: GL(V) \to GL(W)$ where W is a complex vector space.

We'll assume all reps are finite-dimensional, ie. $\dim W < \infty$.

The rep φ is <u>polynomial</u> (resp, rational) if, after choosing ordered bases for V and W,

(resp, rational functions)

the entries of $\varphi(A)$ are polynomials ^ in the entries of $A \in GL(n, \mathbb{C})$.

A rep φ is <u>homogeneous of degree m</u> if $\varphi(\alpha A) = \alpha^m \varphi(A)$ for all $\alpha \in \mathbb{C}^* = \mathbb{C} - \{0\}$.
If φ is a polynomial (or rational) rep, this condition is equiv. to saying each entry of $\varphi(A)$ is a homog. poly of deg m.

<u>Example:</u> $n=2$. Define $\varphi: GL(2,\mathbb{C}) \to GL(3,\mathbb{C})$ by

$$\varphi\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{bmatrix}.$$

Can check φ is a group h'ism.
Entries on RHS are homog. polys of degree 2, so φ is a homog. poly rep of dim 3 & degree 2.

<u>More Examples:</u> $(A \in GL(n,\mathbb{C}))$

- $\varphi(A) = 1 \in \mathbb{C}$ $\forall$ $A \in GL(n,\mathbb{C})$.
  The <u>trivial rep</u>. Homog poly rep of dim 1 & degree 0.

- $\varphi(A) = A$ $\forall$ A. The <u>defining rep</u>.
  Homog. poly rep of dim n & degree 1.

- $\varphi(A) = (\det A)^m$ where $m \in \mathbb{Z}$.
  If $m \geq 0$, this is a homog poly rep of dim 1 & degree mn. If $m < 0$, φ is rational not polynomial. Deg still mn.
- $\varphi(A) = |\det A|^{\sqrt{2}}$. Not a rational rep.
- $\varphi(A) = A^{-1}$. Not a rep.
- $\varphi(A) = \overline{A}$ (complex conj. of A).
  Nonrational rep of dim n, not homog.