155 Lecture 20

Recall that we define a 3-dim Young diagram (or plane partition) to be a 2D array of pos. #'s s.t. both rows + columns weakly decrease + the array of #'s lies in a Young diagram

If T is a plane partition whose entries add to n, we say T is a plane partition of n.

Let $p_{r,s,t}(n)$ = # plane partitions of n which fit inside an $r \times s \times t$ box.

Theorem (MacMahon):

$$\sum_{n \geq 0} p_{r,s,t}(n)\, q^n = \prod_{i=1}^{r} \prod_{k=1}^{t} \frac{1-q^{i+k+s-1}}{1-q^{i+k-1}}$$

We will use Schur functions.

Consider the semistd tableau

$$\begin{matrix} 1 & 1 & 3 & 3 & 4 \\ 2 & 3 & 4 & & \\ 3 & 5 & 5 & & \\ 4 & 6 & & & \\ 5 & & & & \\ 6 & & & & \end{matrix} \quad \longleftrightarrow \quad x_1^2 x_2 x_3^4 x_4^3 x_5^3 x_6^2$$

We can map this tableau to a plane partition where

1	represents a stack of	6 boxes
2	" "	5 boxes
3	" "	4 "
4	" "	3 "
5	" "	2 "
6	" "	1 "

So if we send
$$x_1 \mapsto q^6$$
$$x_2 \mapsto q^5$$
$$\vdots$$
$$x_6 \mapsto q$$ then

the corresponding monomial
$$x_1^2 x_2 x_3^4 x_4^3 x_5^3 x_6^2 \mapsto q^{50}$$

where $50 =$ total # boxes in plane partition.

Therefore $S_{(5,3,3,2,1,1)}(q^6, q^5, q^4, q^3, q^2, q)$ is the gen. function for <u>column strict</u> plane partitions w/ all stack heights ≤ 6 & w/ row lengths given by the parts of λ: 5, 3, 3, 2, 1, 1.

Proof of Theorem: Let $\lambda = s^r$ be the partition w/ r copies of s.

$s_\lambda(q^{t+r}, q^{t+r-1}, \dots q)$ is the gen. function for column strict plane partitions w/ exactly r rows, each of length s, w/ the largest stack of height $\leq t+r$.

We want to get rid of "column strict" condition : If we remove one cube from each stack in row r, 2 cubes from each stack in row $r-1$, ... r cubes from each stack in row 1, we will get exactly what we want:

a plane partition contained in $r \times s \times t$ box.

Each such plane partition can be obtained uniquely like this so we are interested in:

$$q^{-sr(r+1)/2} \quad s_\lambda(q^{t+r}, \dots, q) \quad \text{where}$$

$$\lambda = \begin{cases} s & \text{if } 1 \leq i \leq r \\ 0 & \text{if } r < i \leq t+r \end{cases}$$

Recall that s_λ is a quotient of determinants.

So $s_\lambda(q^{t+r}, \dots, q) = X/Y$ where

$$X = \det\left(\left(q^{t+r-j+1}\right)^{t+r-i+\lambda_i}\right)_{i,j=1}^{t+r}$$

$$Y = \prod_{1 \le i < j \le t+r} \left(q^{t+r-i+1} - q^{t+r-j+1}\right).$$

First simplify X:

For each i, take common factor of $q^{t+r-i+\lambda_i}$ from the ith row.

$$\sum_{i=1}^{t+r} t+r-i+\lambda_i = (t+r)^2 - \frac{(t+r)(t+r+1)}{2} + sr = \frac{(t+r)}{2}(t+r-1) + sr$$

So $X = q^{sr+(t+r)(t+r-1)/2} \det\left(q^{(t+r-j)(t+r-i+\lambda_i)}\right)$

Using Vandermonde Formula, we get

$$X = q^{sr+(t+r)(t+r-1)/2} \prod_{1 \le i < j \le t+r} \left(q^{t+r-i+\lambda_i} - q^{t+r-j+\lambda_j}\right)$$

Vandermonde

Recall $\begin{vmatrix} x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \\ x_1^{n-2} & x_2^{n-2} & & x_n^{n-2} \\ \vdots & & & \\ x_1 & x_2 & \dots & x_n \\ 1 & 1 & \dots & 1 \end{vmatrix} = \prod_{1 \le i < j \le n} (x_i - x_j)$

Now: Y.

Can remove a q from each of the $(t+r)(t+r-1)/2$ terms in Y.

So $$Y = q^{(t+r)(t+r-1)/2} \prod_{1 \le i < j \le t+r} \left(q^{t+r-i} - q^{t+r-j}\right)$$

Now the gen. function we are interested in is:

$$q^{-sr(r+1)/2} \, s_\lambda\left(q^{t+r}, \ldots, q\right)$$

$$= q^{-sr(r+1)/2} \, \frac{X}{Y}.$$

$$= q^{-sr(r+1)/2} \cdot q^{sr} \prod_{1 \le i < j \le t+r} \frac{q^{t+r-i+\lambda_i} - q^{t+r-j+\lambda_j}}{q^{t+r-i} - q^{t+r-j}}$$

$$= q^{-sr(r-1)/2} \prod_{1 \le i < j \le t+r} \frac{q^{t+r-i+\lambda_i} - q^{t+r-j+\lambda_j}}{q^{t+r-i} - q^{t+r-j}}$$

When i and j are both $\le r$, this term is

$$= \frac{q^{t+r-i+s} - q^{t+r-j+s}}{q^{t+r-i} - q^{t+r-j}} = q^s$$

There are $\binom{r}{2} = \frac{r(r-1)}{2}$ such pairs $i<j$,

so this cancels the $q^{-sr(r-1)/2}$

The only terms left are those where
$1 \le i \le r$ and $r+1 \le j \le r+t$

(If $i>r$ then $j>r \Rightarrow$ the corresp term is 1)

So the gen. function is:

$$\prod_{i=1}^{r} \prod_{j=r+1}^{r+t} \frac{q^{t+r-i+s} - q^{t+r-j}}{q^{t+r-i} - q^{t+r-j}}$$

Multiply num. + denom. of each term by q^{j-t-r}:

$$\prod_{i=1}^{r} \prod_{j=r+1}^{r+t} \frac{1-q^{j-i+s}}{1-q^{j-i}}$$

Replace index j by $k+r$ $(k=j-r)$ + i by $r+1-i$ to get:

$$\prod_{i'=1}^{r} \prod_{k=1}^{t} \frac{1-q^{i'+k+s-1}}{1-q^{i'+k-1}} \quad ✪$$

$\blacksquare$

Corollary (MacMahon's Formula)

$$\sum_{n\geq 0} pp(n)\, q^n = \prod_{i\geq 1} \frac{1}{(1-q^i)^i}.$$

Proof: Let $r,s,t \to \infty$ in $(*)$

Write
$$\frac{1-q^{i+k-1+s}}{1-q^{i+k-1}} = \frac{1-q^{i+k-1+s}}{1-q^{i+k-1+(s-1)}} \cdot \frac{1-q^{i+k-1+(s-1)}}{1-q^{i+k-1+(s-2)}} \cdots \frac{1-q^{i+k-1+1}}{1-q^{i+k-1}}$$

Then RHS of $(*)$ is

$$\prod_{i=1}^{r} \prod_{j=1}^{s} \prod_{k=1}^{t} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}} \qquad \text{So}$$

$$\sum_{n\geq 0} p_{r,s,t}(n)\, q^n \overset{(*2)}{=} \prod_{i=1}^{r} \prod_{j=1}^{s} \prod_{k=1}^{t} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}$$

Let $n = i+j+k-1$ ie. $n+1 = i+j+k$.

That is, (i,j,k) is an <u>ordered</u> partition of $n+1$ into 3 parts.

There are $\binom{n}{2}$ ordered partitions of $n+1$ into 3 parts. Why? ASK!

Write $n+1 = 1+1+1+\dots+1$
$n+1$ copies of 1, with n "dividers".
Choose 2 "dividers" to get ordered partition of $n+1$ (in $\binom{n}{2}$ ways).

So we can rewrite (#2) as: $\sum_{n\geq 0} p_{r,s,t}(n)q^n =$

$$\prod_{n\geq 2} \left(\frac{1-q^n}{1-q^{n-1}}\right)^{\binom{n}{2}} =$$

$$\frac{1-q^2}{1-q} \cdot \frac{(1-q^3)^3}{(1-q^2)^3} \cdot \frac{(1-q^4)^6}{(1-q^3)^6} \cdots \frac{(1-q^n)^{\binom{n}{2}}}{(1-q^{n-1})^{\binom{n}{2}}} \cdot \frac{(1-q^{n+1})^{\binom{n+1}{2}}}{(1-q^n)^{\binom{n+1}{2}}} \cdots$$

Since $\binom{n+1}{2} - \binom{n}{2} = \dfrac{n(n+1)-n(n-1)}{2} = n$, this becomes $\dfrac{1}{1-q} \cdot \dfrac{1}{(1-q^2)^2} \cdots \dfrac{1}{(1-q^n)^n} \cdots$

$$= \prod_{n\geq 1} \frac{1}{(1-q^n)^n}$$

$$\therefore \sum_{n\geq 0} pp(n)\, q^n = \prod_{i\geq 1} \frac{1}{(1-q^i)^i}$$

Corollary: $\displaystyle pp(r\times s\times t) = \prod_{i=1}^{r} \prod_{j=1}^{s} \prod_{k=1}^{t} \frac{i+j+k-1}{i+j+k-2}$

Proof: Divide each num. & denom. here

$$\prod_{i=1}^{r} \prod_{j=1}^{s} \prod_{k=1}^{t} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}} \quad (\#2)$$

by $1-q$.

Let $[a]$ denote $1+q+\dots+q^{a-1} = \frac{1-q^a}{1-q}$.

(Call the q-analog of a.)

So (#2) becomes

$$\sum_{n \geq 0} p_{r,s,t}(n)\, q^n = \prod_{i=1}^{r} \prod_{j=1}^{s} \prod_{k=1}^{t} \frac{[i+j+k-1]}{[i+j+k-2]}.$$

If we plug in $q=1$, we get

$$PP_{r \times s \times t} = \prod_{i=1}^{r} \prod_{j=1}^{s} \prod_{k=1}^{t} \frac{i+j+k-1}{i+j+k-2}$$

$\boxed{\text{B}}$

On the ASM Conj:

It was eventually proved in 1995, first by Zeilberger + then by Kuperberg.

Zeilberger's proof was a very complicated induction which showed that the # of "gog triangles" $G(n,k)$ = # of "magog triangles" $M(n,k)$.

Def: A monotone triangle or gog triangle is a triangular array of #'s (n numbers) on each side w/ entries between 1 & n, w/ strict increase across rows & weak increase as one moves diagonally up or down to the right.

Ex.

$$\begin{array}{ccccccccc} & & & & 4 & & & & \\ & & & 2 & & 5 & & & \\ & & 1 & & 4 & & 5 & & \\ & 1 & & 3 & & 4 & & 5 & \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array}$$

There is an easy bijection of ASM's & monotone triangles.

$$\begin{matrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{matrix} \quad \longleftrightarrow \quad \begin{matrix} & & & & 4 & & & & \\ & & & 2 & & 5 & & & \\ & & 1 & & 4 & & 5 & & \\ & 1 & & 3 & & 4 & & 5 & \\ 1 & & 2 & & 3 & & 4 & & 5 \end{matrix}$$

To get entries of i^{th} row on RHS, add component-wise the first i rows on LHS then record position of the 1's.

eg
$$\begin{array}{r} (0\ 0\ 0\ 1\ 0) \\ +\ (0\ 1\ 0\ -1\ 1) \\ +\ (1\ -1\ 0\ 1\ 0) \\ \hline 1\ 0\ 0\ 1\ 1 \end{array} \longmapsto (1, 4, 5)$$

Def: A magog Δ of order n is a triangular array w/ entries 1 to n that increase weakly across rows + down columns + s.t. all entries in column j are $\leq j$.

Ex. for $n=5$

$$\begin{matrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ 1 & 1 & 1 & 3 & \\ 1 & 1 & 2 & 4 & 5 \end{matrix}$$

(which are in bij w/ totally symmetric self complementary plane partitions)

Magog Δ's of order n were proved (by Andrews) to be enumerated by

$$\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}.$$

Let $M(n,k)$ = # of possible configurations for the bottom k rows of a magog Δ of order n.

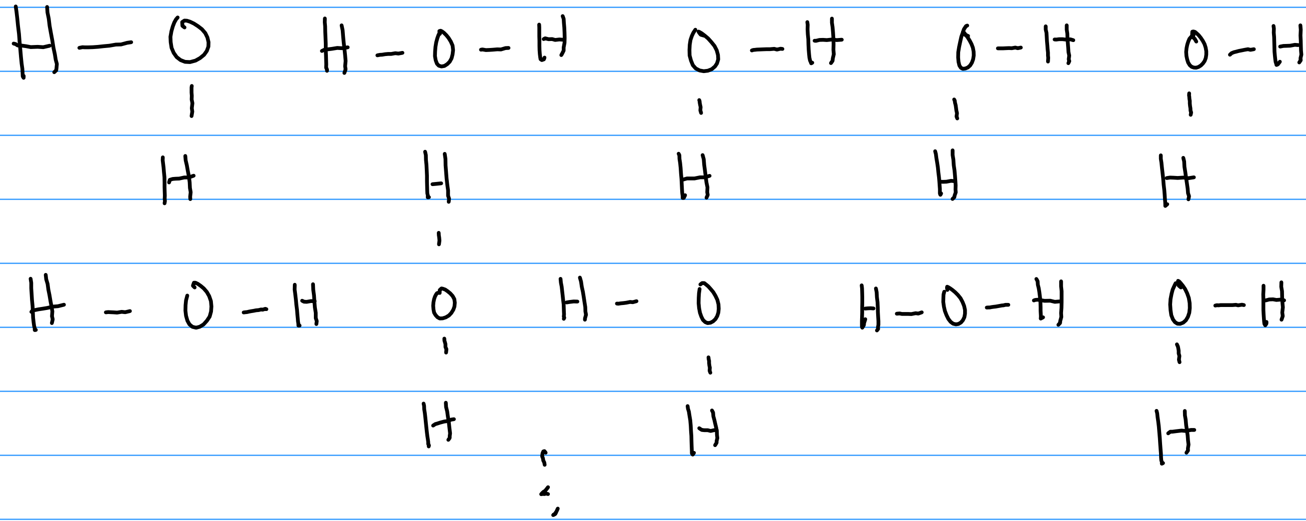
Let $G(n,k)$ = # of possible configuration for the first k diagonals of a gog Δ.

$\therefore$ to prove the ASM Conj, Zeilberger proved $M(n,k) = G(n,k)$ (in particular $M(n,n) = G(n,n)$) + then used Andrews' result. 83-page pf!

Later in the same year ('95), Kuperberg gave a much simpler proof of the ASM Conjecture.

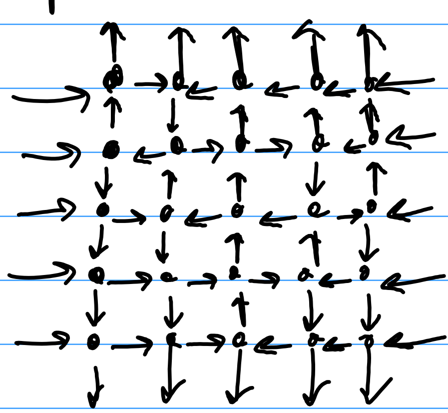
He used techniques from stat. mech.

Physicists had in fact already been studying ASM's, but in disguise — they called their objects square ice. Certain 2D configuration of water molecules.



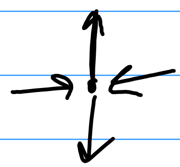
(To convert a config of square ice to an ASM, replace each horiz. molecule w/ a $+1$, each vert. molecule w/ a -1, & all angled molecules w/ 0.)

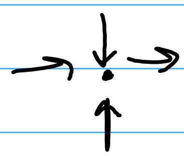
Physicists typically represent a config. of square ice w/ a state of the 6 vertex model — a directed graph on a square lattice in which each vertex has in-degree & out-deg 2, as in:



Called "6-vertex" because the local config around each vertex has 6 possibilities.

These states — when we fix the boundary config shown above — corresp to ASM's, by replacing

 w/ a 1,

 w/ a -1,

+ all other vertices w/ 0's.

Kuperberg's proof involved:

- results of Izergin–Korepin on 6 vertex model
- Yang Baxter equation
- Dodgson condensation
- Symm funcn...

Further reading: Bressoud's book "the story of the alternating sign matrix conj."