Draw Young's lattice?

155 Lecture 8

Recall $S_n^\vee$ is set of equiv. classes of irreps of S_n

Branching graph is directed graph s.t.

- Vertices are elemts of $\bigcup_{n\geq 0} S_n^\vee$.
- we put edge $\mu \xrightarrow{k} \lambda$ $(\mu \in S_{n-1}^\vee, \lambda \in S_n^\vee)$ iff $\mathrm{Res}^{S_n}_{S_{n-1}} V^\lambda$ contains k copies of V^μ

When branching graph is simple (k always 0 or 1), we get canonical basis for V^λ by restricting to S_{n-1}, then S_{n-2}, ... S_1 :

$$V^\lambda = \bigoplus_T V_T \quad \text{indexed by all}$$

possible chains $T = \lambda_0 \nearrow \lambda_1 \nearrow \dots \nearrow \lambda_n = \lambda$ where $\lambda_i \in S_i^\wedge$

Choosing a unit vector v_T in each one-dim v.s. V_T, we get the GT basis.

Main idea: analyze the rep V^λ by looking at how a large comm. subalg of $\mathbb{C}[S_n]$ acts on GT basis — study the eigenvalues.

$Z(n)$: center of $\mathbb{C}[S_n]$.

$GT(n) = \langle Z(1), Z(2), \dots, Z(n) \rangle$ (GT algebra)

$GT(n)$ is commutative.

$$GT(n) \subset \mathbb{C}[S_n] = \bigoplus_{\lambda \in S_n^\vee} \mathrm{End}(V^\lambda) = \begin{pmatrix} \boxed{\ast} & & 0 \\ & \boxed{\ast} & \\ 0 & & \ddots \; \boxed{\ast} \end{pmatrix}$$

(*) <u>Prop</u>: (1) $GT(n)$ is the alg. of diagonal matrices w/ respect to the GT basis in each V^λ

(2) $GT(n)$ a maximal commutative subalg. in $\mathbb{C}[S_n]$

(3) $v \in V^\lambda$ is in the GT basis iff v is a common eigenvector of elmts of $GT(n)$

(4) Each basis element is uniquely determined by eigenvalues of elts of $GT(n)$.

For $i = 1, 2, \dots, n$, define YJM element

$$X_i = (1\ i) + (2\ i) + \dots + (i{-}1\ \ i) \in \mathbb{C}[S_n]$$

$X_i \in GT(n)$ $\forall\ i \le n$.

In particular, the X_i's commute.

<u>Prop</u>: The GT algebra is generated by the YJM elements: $GT(n) = \langle X_1, X_2, \dots, X_n \rangle$.

Last time we almost finished showing that branching is simple.

<u>Def</u>: If $A \supset B$ are algebras, the <u>centralizer</u>

$$Z(A, B) = \{ a \in A \mid ab = ba \quad \forall\ b \in B \}$$

Lemma: $M \subset N$ two algebras. Let V be a f.dim irred. rep of N.

(1) $\mathrm{Res}^N_M V$ has simple multiplicities

(2) $Z(N, M)$ is commutative.

We want to show $\mathrm{Res}^{S_n}_{S_{n-1}} V^\lambda$ has simple multiplicites so need to show

$Z_{n-1,1} := Z(\mathbb{C}[S_n], \mathbb{C}[S_{n-1}])$ is commutative.

Theorem: $Z(n) \subset \langle Z(n-1), X_n \rangle$ (sketched)

Theorem: The branching of the chain $\mathbb{C}[S_1] \subset \dots \subset \mathbb{C}[S_n]$ is simple, i.e. multiplicities of restriction of irreps of $\mathbb{C}[S_n]$ to $\mathbb{C}[S_{n-1}]$ are equal to 0 or 1.

Proof sketch: Need to show the centralizer $Z(n-1, 1) = Z(\mathbb{C}[S_n], \mathbb{C}[S_{n-1}])$ is commutative. We do so by showing that

$$Z(n-1, 1) \subset \langle Z(n-1), X_n \rangle.$$

$Z(n-1) \subset GT(n)$ and $X_n \in GT(n)$ and $GT(n)$ is commutative.

$\therefore$ $\langle Z(n-1), X_n \rangle$ is comm.

Cor: In each irrep of S_n, the GT basis is determined up to scalar factors.

Pf: As we saw earlier, simple branching means we can take an irrep V^λ of S_n & restrict to S_{n-1}, S_{n-2}, etc, each time decomposing into irreps, until we get

$\bigoplus_T V^T$ where we sum over all

$T = \lambda^0 \nearrow \lambda^1 \nearrow \cdots \nearrow \lambda^n = \lambda$

We define the <u>Young basis</u> to be the union of all GT bases for all irreps V^λ of S_n.

Since the YJM elemts $X_k = \sum_{i=1}^{k-1} (i\ k) \in \mathbb{C}[S_n]$ $(k = 1 \dots n)$ generate the GT-algebra (which is diagonal w/respect to Young basis), the Young basis consists of common eigenvectors of the X_i's.
(each V_T is an eigenvector for all the X_i)
+ there's no other common eigenvector

Let $\alpha(v) = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$ be the eigenvalues of $X_1, \dots, X_n$ on v.
We call $\alpha(v)$ the <u>weight</u> of v.

Let $\mathrm{Spec}(n) = \{\alpha(v) \mid v \text{ is in the GT basis}\}$

By Prop Ⓐ, a point $\alpha(v) \in \mathrm{Spec}(n)$ determines v up to a scalar factor.

$\therefore \quad |\mathrm{Spec}(n)| = \sum_{\lambda \in S_n^\vee} \dim V^\lambda$

By def. of Young basis,

$\mathrm{Spec}(n) \leftrightarrow$ set of all paths $\lambda^0 \nearrow \lambda^1 \nearrow \dots \nearrow \lambda$ in the branching graph where $\lambda \in S_n^\vee$

Denote bijection by

$\alpha \mapsto T_\alpha$ $\qquad$ $T \mapsto \alpha(T)$

Let v_α denote the vector (unique up to scalar mult) of the Young basis corresp. to weight α.

Equiv. relation $\sim$ on $\mathrm{Spec}(n)$:
we say $\alpha \sim \beta$ for $\alpha, \beta \in \mathrm{Spec}(n)$ if v_α and v_β belong to the same irred. S_n-module, or equiv, if the paths T_α and T_β have the same endpoint.

Clearly $|\mathrm{Spec}(n)/\sim| = |S_n^\vee|$

Plan: (1) Describe $\mathrm{Spec}(n)$
(2) Describe $\sim$

Background on S_n

Let s_i denote $(i \;\; i+1)$

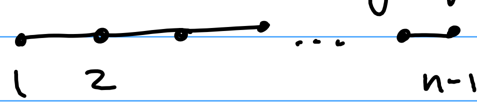
Theorem: S_n is generated by $\{s_i \mid 1 \le i \le n-1\}$ w/ the relations $s_i^2 = 1$, $s_i s_j = s_j s_i$ for $|j-i| \geq 2$, and $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.

Proof well-known. Topic for project?

Remark:
S_n is an example of a *Coxeter group*.
Given any simply laced graph G (ie. edges do not have multiplicities), say on vertices $\{1, .., n\}$, we can define a group w/ generators $\{s_i \mid 1 \le i \le n\}$ and the relations:

$$s_i^2 = 1 \quad \forall i$$
$$(s_i s_j)^2 = 1 \quad \text{if } i \,\&\, j \text{ not adjacent in } G$$
$$(s_i s_j)^3 = 1 \quad \text{if } i \,\&\, j \text{ adj. in graph}$$

Ask: S_n is the Coxeter group for which graph?



Towards a description of Spec(n):

Def: We say that $\alpha = (a_1, \dots, a_n)$ is a content vector, $\alpha \in \mathrm{Cont}(n)$, if α satisfies the following conditions:

(1) $a_1 = 0$

(2) $\{a_q - 1, a_q + 1\} \cap \{a_1, .., a_{q-1}\} \neq \emptyset$ for all $q > 1$

(3) If $a_p = a_q = a$ for some $p < q$ then $\{a-1, a+1\} \subset \{a_{p+1}, \dots, a_{q-1}\}$

(ie. between 2 occurrences of a in a content vector there should also be occurrences of $a-1$ and $a+1$)

Clearly $\mathrm{Cont}(n) \subset \mathbb{Z}^n$.

Theorem: $\mathrm{Spec}(n) = \mathrm{Cont}(n)$, where $\mathrm{Spec}(n)$ is the spectrum of $GT(n)$ w/ respect to the YJM-generators $X_1, \dots, X_n$.

(defer proof)

Def: Consider a Young diagram λ. Given a box $\square \in \lambda$, the number
$c(\square) := x\text{-coord of } \square - y\text{-coord of } \square$
is called the <u>content</u> of λ.

Eg

	1	2	3	4	5
1	0	1	2	3	4
2	-1	0	1	2	
3	-2	-1			

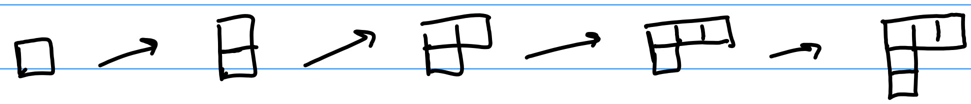
Recall: we have edge $\nu \to \eta$ in Young's lattice $\mathbb{Y}$ iff two Young diagrams ν and η satisfy $\nu \subset \eta$ and $\eta - \nu$ is a single box. In this case we write $\nu \nearrow \eta$.

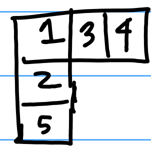
Let $\mathrm{Tab}(\nu)$ denote the set of paths in $\mathbb{Y}$ from $\emptyset$ to ν. Such paths are called std tableaux, because:

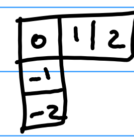
a convenient way to represent a path T in $\mathbb{Y}$ from $\emptyset$ to ν
$\emptyset = \nu_0 \nearrow \dots \nearrow \nu_n = \nu$ is to write the numbers $1, \dots, n$ in the boxes $\nu_1 - \nu_0, \nu_2 - \nu_1, \dots, \nu_n - \nu_{n-1}$.

Let $\mathrm{Tab}(n) = \bigcup_{|\nu| = n} \mathrm{Tab}(\nu)$

Prop: Let $T = \nu_0 \to \dots \to \nu_n \in \mathrm{Tab}(n)$.
The mapping $T \mapsto (c(\nu_1 - \nu_0), \dots, c(\nu_n - \nu_{n-1}))$ is a bijection of the set of tableaux $\mathrm{Tab}(n)$ + the set of content vectors $\mathrm{Cont}(n)$ defined earlier.

Ex: 

Tableaux rep. by 

$c(T) = (0, -1, 1, 2, -2)$ because  are the contents

Check that this satisfies!

(1) $a_1 = 0$
(2) $\{a_q - 1, a_q + 1\} \cap \{a_1, \dots, a_{q-1}\} \neq \emptyset$ for all $q > 1$
(3) If $a_p = a_q = a$ for some $p < q$ then $\{a-1, a+1\} \subset \{a_{p+1}, \dots, a_{q-1}\}$
(i.e. between 2 occurrences of a in a content vector there should also be occurrences of $a-1$ and $a+1$)

Proof: Clearly any content vector of a tableaux satisfies these 3 conditions. And these conditions will uniquely determine a tableaux as a sequence of boxes.

Punchline: Spec(n) = the set of content vectors of std tableaux w/ n boxes
So this is how we could have "discovered" std tableaux:

- Using fact that $\mathrm{Res}^{S_n}_{S_{n-1}} V$ of an irrep V has simple multiplicities, inductively construct "natural" basis v_T for $\bigoplus_{\lambda \in S_n^\vee} V^\lambda$
- Define the GT(n) algebra, a maximal comm. subalgebra of $\mathbb{C}[S_n]$ which is diagonal matrices w/ respect to the v_T
- Find nice generators $\{X_i \mid 1 \le i \le n\}$ for GT(n) & compute their eigenvalues on the $\{v_T\}$
- Realize that the collection of eigenvalues that the $\{X_i\}$ have on v_T can be described combinatorially as the content vector of some std tableaux! ✓

Example: $\lambda = \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & \\ \cline{1-2} \\ \cline{1-1}\end{array}$. Basis $\leftrightarrow$ std tableaux

Take $T = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 6 & 7 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$ then since $\mathrm{cont}(\lambda) = \begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 3 \\ \hline -1 & 0 \\ \cline{1-2} -2 \\ \cline{1-1}\end{array}$

we get content vector

$$\alpha(T) = (0,-1,1,-2,0,2,3).$$

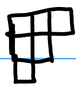
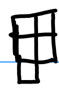
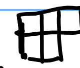
This says that

$$X_1 v_T = 0 v_T$$
$$X_2 v_T = -v_T$$
$$X_3 v_T = v_T$$
$$X_4 v_T = -2v_T$$
$$X_5 v_T = 0$$
$$X_6 v_T = 2v_T$$
$$X_7 v_T = 3v_T$$

Main Theorem: The Young graph $\mathbb{Y}$ is the branching graph of the symm group. The spectrum of the Gelfand Tsetlin algebra $GZ(n)$ $\mathrm{Spec}(n)$ is in bijection w/ the set of paths from $\emptyset$ to a Young diagram λ w/

n boxes. $\mathrm{Spec}(n) = \mathrm{Cont}(n)$, where $\mathrm{Spec}(n)$ is the spectrum of $GZ(n)$ w/ respect to the YJM generators $X_1, \ldots X_n$ & $\mathrm{Cont}(n)$ is the set of content vectors.

<u>Cor 1:</u> We know how to compute $\mathrm{Res}^{S_n}_{S_{n-1}} V^\lambda$:

<u>Ex:</u> $\mathrm{Res}^{S_6}_{S_5} V$ $= V$ $\oplus\ V$ $\oplus\ V$

<u>Cor 2:</u> $\dim V^\lambda = \#$ std tableaux of shape λ

Now, need to go back + prove this:

<u>Theorem:</u> $\mathrm{Spec}(n) = \mathrm{Cont}(n)$, where $\mathrm{Spec}(n)$ is the spectrum of $GT(n)$ w/ respect to the YJM-generators $X_1, \ldots, X_n$.

To do so, need to understand action of S_n on v_T.

<u>Prop:</u> For any vector v_T, $T = \lambda_0 \nearrow \ldots \nearrow \lambda_n$, $\lambda_i \in \hat{S_i}$, and any $k = 1, \ldots, n-1$, the vector $s_k \cdot v_T$ is a linear combination of the vectors

$(k\ k+1)$ $\quad v_{T'}$, $T' = \lambda_0' \nearrow \ldots \nearrow \lambda_n'$ $(\lambda_i' \in \hat{S_i})$

s.t. $\lambda_i' = \lambda_i$, $i \neq k$.

That is, the action of s_k affects only the kth level of the branching graph.

Proof: Let $i > k$. Since $s_k \in S_i$ & the module $\mathbb{C}[S_i] \cdot v_T$ is irred, we have

(※) $\mathbb{C}[S_i] s_k \cdot v_T = \mathbb{C}[S_i] \cdot v_T = V^{\lambda_i}$, where

V^{λ_i} is the irred S_i-module indexed by $\lambda_i \in \hat{S_i}$.

For $i < k$, s_k commutes w/ S_i so (※) holds for $i < k$ also.

Let $T = \lambda_0 \nearrow \ldots \nearrow \lambda_i \nearrow \ldots \nearrow \lambda_n$.

Recall that $\mathbb{C}[S_i].v_T$ (for $i = 1, 2, \ldots, n$) $= V^{\lambda_i}$ ∴ result follows.

(this gives a method to reconstruct the λ^i's by seeing what $\mathbb{C}[S_i].v_T$ is)

∎

Can similarly show that the coeff's of this linear comb. depend only on $\lambda_{k-1}, \lambda_k, \lambda_k', \lambda_{k+1}$.

So action of s_k affects only the k^{th} level + depends only on levels $k-1, k, k+1$ of branching graph. "local"