MATH 155 PROBLEM SET 2 (DUE THURSDAY OCT. 11)

(1) Let T be a numbering of a Young diagram, and recall from class the notion of the row group $R(T)$, the column group $C(T)$, and the Young symmetrizers a_T and b_T.
   (a) For $p \in R(T)$ and $q \in C(T)$, show that $p\, a_T = a_T\, p = a_T$ and $q\, b_T = b_T\, q = \operatorname{sgn}(q) b_T$.
   (b) Show that $a_T\, a_T = \#R(T) a_T$ and $b_T\, b_T = \#C(T) b_T$. (All of the above are computations in the group algebra of the symmetric group.)

(2) Let T be a numbering of a Young diagram with n boxes. Recall that $v_T := b_T \cdot \{T\}$ is an element of M^λ, the S_n-module whose basis is the set of tabloids of shape λ. Show that for any $\sigma \in S_n$, $\sigma \cdot v_T = v_{\sigma \cdot T}$.

(3) Let λ be the partition $(n-1, 1)$ and describe the Specht module S^λ. Have you seen this representation before (with a different name)?

(4) Prove the Erdos-Szekeres theorem: given any $\pi \in S_{nm+1}$, then π contains either an increasing subsequence of length $n+1$ or a decreasing subsequence of length $m+1$.

(5) An involution is a permutation π such that π^2 is the identity. Prove the following facts about involutions:
   (a) π is an involution if and only if $P(\pi) = Q(\pi)$. Use this to find a formula for the number of involutions in S_n in terms of the quantities f^λ.
   (b) The number of fixed points in an involution π is the number of columns of odd length in $P(\pi)$. Hint: use Viennot's shadow diagrams.
   (c) We have

$$1 \cdot 3 \cdot 5 \ldots (2n-1) = \sum_{\lambda \vdash 2n, \lambda' \text{ even}} f^\lambda,$$

   where λ' even means that the conjugate (transpose) of λ has only even parts.

(6) A *two-person ballot sequence* is a permutation $\pi = x_1 x_2 \ldots x_{2n}$ of n ones and n twos such that, for any prefix $\pi_k = x_1 x_2 \ldots x_k$, the number of ones in x_k is at least as great as the number of twos. The nth *Catalan number* C_n is the number of such sequences.

(a) Prove the recurrence $C_{n+1} = C_nC_0 + C_{n-1}C_1 + \cdots + C_0C_n$ for $n \geq 0$.

(b) The Catalan numbers also count the following sets. Show this in two ways: by verifying that the recurrence is satisfied, and by giving a bijection with a set of objects already known to be counted by the C_n.
- standard tableaux of shape (n, n)
- permutations $\pi \in S_n$ with longest decreasing subsequence of length at most two.

(c) Use the connection to tableaux to show that $C_n = \frac{1}{n+1}\binom{2n}{n}$

(7) For $\lambda = (\lambda_1, \dots, \lambda_r)$ a partition of n and for $1 \leq i \leq r$, let $f(\lambda, -i)$ denote the number of standard tableaux with shape $(\lambda_1, \dots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \dots, \lambda_r)$. Similarly let $f(\lambda, +i)$ denote the number of standard tableaux with shape $(\lambda_1, \dots, \lambda_{i-1}, \lambda_i + 1, \lambda_{i+1}, \dots, \lambda_r)$. If the resulting shape is not a partition, we say $f(\lambda, +i)$ or $f(\lambda, -i)$ is 0. Use these notions to give an alternative proof that $n! = \sum_\lambda (f^\lambda)^2$. Hint: first show the following:
- $f^\lambda = \sum_{i=1}^{r} f(\lambda, -i)$
- $(n+1)f^\lambda = \sum_{i=1}^{r+1} f(\lambda, +i)$

(8) Prove that, up to sign, the determinant of the character table for S_n is
$$\prod_{\lambda \vdash n} \prod_{\lambda_i \in \lambda} \lambda_i.$$
Here λ_i denotes the length of the ith row of λ.