Math 191 Solution Set 2

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II.35 This is just the hypergeometric distribution. If there are to be i aces among r cards, then we choose i out of the 4 aces and $r-i$ out of the 48 non-aces. The sample space consists of the r-card subsets of 52 cards. Hence

$$p_i(r) = \frac{\binom{4}{i}\binom{48}{r-i}}{\binom{52}{r}}, i = 0, \ldots, 4.$$

In particular,

$$p_4(52-r) = \frac{\binom{4}{4}\binom{48}{52-r-4}}{\binom{52}{52-r}} = \frac{\binom{4}{4}\binom{48}{48-r}}{\binom{52}{52-r}} = \frac{\binom{4}{0}\binom{48}{r}}{\binom{52}{r}} = p_0(r).$$

II.42 A deck of cards has 4 aces and 48 non-aces. We can consider the aces and non-aces to be indistinguishable, respectively, because permuting the locations of aces and of non-aces does not affect whether two aces are adjacent. Hence the sample space S, all the arrangements of 4 indistinguishable aces and 48 indistinguishable non-aces, has size $\binom{52}{4}$.

One way to arrange the cards is to insert the 4 aces into a deck of 48 non-aces. Because the aces are to be nonadjacent, only one ace can be placed between each pair of consecutive non-aces. Hence we can consider the "gaps" between the 48 non-aces (and before the first non-ace and after the last non-ace) to be bins, and our task is to place the 4 aces into 4 different bins. Hence we are just choosing 4 bins out of 49 bins. Thus the number of arrangements of non-adjacent aces is $\binom{49}{4}$, and the probability is $\binom{49}{4}/\binom{52}{4}$.

Extra For part a, the sample space is the set of 13-card subset of 52 cards, which has size $\binom{52}{13}$. To get a bridge hand with $(4,4,3,2)$ distribution, we first determine which suit gets which number of cards. Two suits get 4 cards, 1 gets 3 cards, and 1 gets 2 cards; so this first step has $\binom{4}{2,1,1}$ possibilities. After determining the suits, we need to pick 4 cards from each of the first 2 suits, 3 cards from the third suit, and 2 cards from the fourth suit. This is just the hypergeometric distribution; the number of ways to do this is $\binom{13}{4}\binom{13}{4}\binom{13}{3}\binom{13}{2}$. Hence the probability of a $(4,4,3,2)$ distribution is

$$\frac{\binom{4}{2,1,1}\binom{13}{4}\binom{13}{4}\binom{13}{3}\binom{13}{2}}{\binom{52}{13}} \approx 0.216.$$

Similarly, the probabilities of a $(4,3,3,3)$, $(5,3,3,2)$, or $(5,4,3,1)$ distribution are respectively

$$\frac{\binom{4}{3,1}\binom{13}{4}\binom{13}{3}\binom{13}{3}\binom{13}{3}}{\binom{52}{13}} \approx 0.105, \frac{\binom{4}{2,1,1}\binom{13}{5}\binom{13}{3}\binom{13}{3}\binom{13}{2}}{\binom{52}{13}} \approx 0.155,$$

$$\frac{\binom{4}{1,1,1,1}\binom{13}{5}\binom{13}{4}\binom{13}{3}\binom{13}{1}}{\binom{52}{13}} \approx 0.129.$$

For part b, we also need to determine which suit gets which number of cards, same as in part a. Then we need to determine the order of the suits among the 13 die rolls. For the $(4,4,3,2)$ distribution, the number of ways of arranging the suit orders is $\binom{13}{4,4,3,2}$. Actually getting a particular sequence of 13 die rolls is an occurrence with probability $\left(\frac{1}{4}\right)^{13}$. Hence the probability of a $(4,4,3,2)$ distribution is $\frac{1}{4^{13}}\binom{4}{2,1,1}\binom{13}{4,4,3,2} \approx 0.161$. Similarly, the probabilities for a $(4,3,3,3)$, $(5,3,3,2)$, or $(5,4,3,1)$ distribution are respectively $\frac{1}{4^{13}}\binom{4}{3,1}\binom{13}{4,3,3,3} \approx 0.071$. $\frac{1}{4^{13}}\binom{4}{2,1,1}\binom{13}{5,3,3,2} \approx 0.129$, and $\frac{1}{4^{13}}\binom{4}{1,1,1,1}\binom{13}{5,4,3,1} \approx 0.129$. Each probability is smaller than the corresponding probability in part a.