MATHEMATICS 191, FALL 2003
MATHEMATICAL PROBABILITY
Assignment #6

<u>Problems to be discussed in section:</u> All problems are from Grimmett and Stirzaker, 1000 Exercises in Probability. The solutions are all in the book!

Add together the number of letters in your first and last name. If the sum is odd, prepare problems 1,3, and 5. If it is even, prepare 2, 4, and 6.

1. Section 3.9, problem 5.
2. Section 3.10 , problem 3.
3. Section 3.11 , problem 39(a).
4. Section 3.11, problem 28.
5. Section 3.11, problem 24 (first new solution).
6. Section 3.11, problem 24 (second new solution).

<u>Problems to be handed in on Thursday, October 30:</u>

1. A convenience store offers Lottery scratch tickets for $1, $2, or $4. When you scratch the ticket, with probability $p = \frac{1}{2}$ you receive double the cost of the ticket, otherwise you receive nothing. So buying a sequence of scratch tickets is a random walk.

   You enter the store with $8, determined to keep buying tickets until you either achieve your target of $12 or run out of money. Using the formulas derived in class and in the textbook, calculate the probability of achieving your target for the three cases of tickets that cost $1, $2, or $4.

   Repeat the calculation for the more realistic assumption that $p = \frac{1}{3}$.

2. Harvard and Yale's football teams are so evenly matched the the probability of Harvard winning any game is exactly $p = \frac{1}{2}$. The teams agree to keep playing a series of games until one team has won three games more than the other. What is the probability that each team will have been ahead in the series before it ends? The analysis of problem 3.11.32 in the text may be useful.

3. In class the ballot theorem was proved by a combinatorial approach using the reflection principle, but a recurrence approach is also possible. Suppose that in an election candidate George has m votes and candidate Al has n votes, where $m > n$, and the ballots are arranged in a random order and recounted publicly. Let $p(m, n)$ denote the probability that George is ahead at every stage during the recount. There are two ways

for this to happen: either the last ballot counted is for George, or it is for Al. By conditioning on this last ballot (which is more likely to be for George), set up a recurrence for $p(m, n)$ and show that it is satisfied by $p(m, n) = \frac{m-n}{m+n}$ in accordance with the ballot theorem.

4. An impoverished student starts with a balance of \$0 in his bank account, which has a credit line so that the balance can go negative. Each day he either deposits \$1 with probability $\frac{1}{2}$ or withdraws \$1 with probability $\frac{1}{2}$. Show that the probability that after $2n$ days his balance is again \$0, without the credit line having been used, is

$$\frac{(2n)!}{(n+1)(n!)^2}.$$

(Hint: after 1 day the balance must be +\$1. Any path that subsequently drops below \$0 matches a reflected path that has a balance of -\$3 after the first day.

5. Do a geometric proof of the theorem that for for a random walk of $2n$ steps starting at 0, the number of paths that end at 0 is equal to the number of paths that never fall below 0 during the first $2n$ steps. Here is the basic idea, attributed by W. Feller to E. Nelson.

   Consider a path that starts and ends in level 0. Let the first occurrence of the (global) minimum of this path be in level $-m$ after k steps. Reflect the portion of the path from 0 to k in the vertical line for time k, then slide it $2n - k$ units to the right and m levels up so that you can attach it to the right of $2n$. Show that this creates a path of $2n$ steps (with a different origin at time k, level $-m$) that never falls below its starting level and that the construction can be inverted to establish a bijection between the two types of paths.

   For the case where Harvard and Yale play 6 games, there are 6 sequences (like YHHY) that lead to a tied series after 4 games and 6 sequences (like HYHH) that have the property that Harvard is never behind in the series. Exhibit the one-to-one correspondence established by your construction.