MATHEMATICS 191, FALL 2003
MATHEMATICAL PROBABILITY
Assignment #9

Problems to be discussed in section: All problems are from Grimmett and Stirzaker, 1000 Exercises in Probability. The solutions are all in the book!

Add together the number of letters in your first and last name. If the sum is odd, prepare problems 1,3, and 5. If it is even, prepare 2, 4, and 6.

1. Section 4.7, problem 1. The “arguing more directly” solution is very easy if you reduce it to calculating the volume of a solid whose top bounding surface is the graph of $z = \sqrt{xy}$. This gives the probability for the complement of the desired event.
2. Section 4.8, problem 3. Be careful with the limits of integration.
3. Section 4.13, problem 8.
4. Section 4.14, problem 8a.
5. Section 4.13, problem 9.
6. Section 4.13, problem 14.

Problems to be handed in on Thursday, December 4:

As usual, these are all newly invented. Send email if you suspect that you have found a bug in any of them.

1. (a) By using the change of variables

   $$Z = X + Y, V = X - \alpha Y$$

   for arbitrary $\alpha \geq 0$, derive a convolution formula for the density function $f_Z(z)$ that includes as special cases both the formula in the textbook and the alternative one derived in lecture.

   (b) By choosing $\alpha = \sigma^2$, show that if X is $N[0, \sigma^2]$ and Y is $N[0, 1]$, than $Z = X + Y$ is $N[0, \sigma^2 + 1]$.

2. Choose a “random point” within the unit circle by first choosing a random chord, then choosing a point at random (uniform density) on the chord. Find the expectation of the random variable $X^2 + Y^2$, the square of the distance of the point from the center of the circle. Solve the problem for each of the three ways of choosing a ”random chord” in Bertrand’s paradox.

3. Two archeological teams both want to excavate along the same kilometer of road in Iraq. The local authorities lease a supercomputer from Halliburton and use it to generate four independent random variables X_1, X_2, X_3, X_4, all uniformly distributed in [0,1]. The first team is allowed to dig in $[X_1, X_2]$; the second team is allowed to dig in $[X_3, X_4]$. Calculate the probability that their sites do not overlap, using two different methods, as follows:

   (a) Let $U = \max(X_1, X_2)$. Find a distribution function $F_U(u)$ and a density function $f_U(u)$. An easy way to get $F_U(u)$ is to note that the point (X_1, X_2) would have to lie in a certain square if $U \leq u$.

   (b) Let $V = \min(X_3, X_4)$. Find the probability $\mathbb{P}(V > u)$.

   (c) By conditioning on the value of U, find $\mathbb{P}(U \leq V)$. Double this to get the answer.

   (d) Alternatively, just note that if you arrange x_1, x_2, x_3, and x_4, all permutations are equally likely. Count the number of permutations that lead to no overlap.

4. Two points, chosen independently at random inside the unit square with uniform density, are taken as opposite corners of a rectangle with horizontal and vertical sides.

   (a) What is the expectation of the area of the rectangle?

   (b) What is the probability that two such rectangles, chosen independently, do not overlap? (You can reuse a result about nonoverlapping intervals from the preceding problem.)

   (c) Using affine transformations, state generalizations of these results to parallelograms. The proof would be trivial, so you need not write it out.

5. There is a slightly simpler way of finishing the proof of (7) on p. 138 of G&S. Consider a triangle whose vertices are $A = (0, 0)$, $B = (x, 0)$, $C = (0, 1)$. Show that the same approach as in the text leads to the differential equation

$$\frac{dA}{dx} + \frac{3A}{x} = \frac{1}{6}$$

   and solve this equation with boundary condition $A(0) = 0$, perhaps by noticing that x^3 is an integrating factor.