Math 191 Notes, 2003 October 14

Matching Problem, Review of specific case

Matching problem, general case with n letters. Let the probability that the $r = 0$ (number of correct letters is zero).

$$PP(r = 0) = \frac{1}{n!} \sum_{\pi} (1 - I_1) \cdots (1 - I_n)$$

For any n, the probability that any specific set of n letters end up in their correct envelopes is the same as the probability for any other set of n letters.

$$PP(r = 0) = \frac{1}{n!} \sum_{s} (-1)^s \binom{n}{s} (n - s)! I_1 \cdots I_s$$

Thus we get that general formula from last time.

Matching Problem, General case

Let X be the random variable defining the number of "good" letters. What is $PP(X = r)$? Pick the r that will be correct. Then make all the the rest $(n - r)$ incorrect.

$$PP(X = r) = \frac{1}{n!}(n - r)! \sum_{s=0}^{n-r} (-1)^s \frac{1}{s!} = \frac{1}{r!} \sum_{s=0}^{n-r} \frac{(-1)^s}{s!}$$

As $n \to \infty$, this is a poisson distribution with $\lambda = 1$. This is one of many ways of getting that distribution.

"Probabilistic method"

Say we have 17 fenceposts in a circle. 5 of them are rotten. Prove there is at least one set of seven consecutive posts of which 3 are rotten.

Solution using probabilistic concepts: Let R_k be the number of rotten posts in $\{k+1, k+2, \cdots, k+7 \mod 17\}$.

$$EE(R_k) = \sum_{i=1}^{7} I_{k+i} = 7 \cdot \frac{5}{17} = \frac{35}{17} > 2$$

Where I_k is the indicator function for post k being rotten. Since the probability is greater than two, we know there must be event that has three fence posts.

Binomial → Poisson

First we want to prove that Poisson is limiting case of Binomial distribution. Binomial distribution:

$$f(k) = \binom{n}{k} p^k q^{n-k} = \frac{n!}{(n-k)!k!}(\frac{p}{q})^k q^n$$

Now let $p \to 0, n \to \infty$ hold $np = \lambda$.

$$f(k) = (\frac{np}{q}^k \frac{1}{k!} \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n = \frac{\lambda^k}{k!} e^{-\lambda}$$

Aside: for any specified value of k, we can probably do what we were hoping and take the limit as $n \to \infty$. Also, here's a differentiation trick.

$$\int_0^\infty e^{-kx} dx = \frac{1}{k}$$

$$\begin{aligned} \int_0^\infty x^2 e^{-kx} dx &= \int_0^\infty \frac{d^2}{dk^2} e^{-kx} dx \\ &= \frac{d^2}{dk^2}\left(\frac{1}{k}\right) \\ &= \frac{2}{k^3} \end{aligned}$$

Expectation for binomial distribution

$$\begin{aligned} EE(k) = \sum_{k=0}^{n} k f(k) &= \sum k \binom{n}{k} p^k q^{n-k} \\ &= \sum_{k=0}^{n} p \frac{\partial}{\partial p}\left[\binom{n}{k} p^k q^{n-k}\right] \\ &= p\frac{\partial}{\partial p} \sum \binom{n}{k} p^k q^{n-k} \\ &= p\frac{\partial}{\partial p}(p+q)^n \\ &= pn(p+q)^{n-1} = np \end{aligned}$$

And take the limit to get Poisson $np = \lambda$.

Variance for binomial distribution

$$var(X) = EE(X^2) - (E(X))^2$$

$$
\begin{aligned}
EE(X^2) &= \sum_{k=0}^{n} k^2 f(k) \\
&= \sum_{k=0}^{n} k^2 \binom{n}{k} p^k q^{n-k} \\
&= p\partial/\partial p \left(p\partial/\partial p \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \right) \\
&= p\partial/\partial p p\partial/\partial p (p+q)^n \\
&= p\partial/\partial p (pn(p+q)^{n-1}) \\
&= p\left[n(p+q)^{n-1} + pn(n-1)(p+q)^{n-2}\right] \\
&= np + n(n-1)p^2
\end{aligned}
$$

$$var(X) = np + n^2p^2 - np^2 - n^2p^2 = np(1-p) = npq$$

For the Poisson, we start with binomial and $np \to \lambda, p \to 0$: $var(X) = \lambda$.

Geometric Distribution

"Geometric" distribution: How many flips k do I need to see the first head?

$$f(k) = q^{k-1}p$$

$$\sum_{1}^{\infty} f(k) = p \sum_{k=1}^{\infty} q^{k-1} = \frac{p}{1-q} = 1$$

Expectation:

$$EE(X) = \sum_{1}^{\infty} kf(k) = p \sum_{k=1}^{\infty} kq^{k-1} = p\frac{d}{dq}(\sum_{k=0}^{\infty} q^k) = p\frac{d}{dq}\frac{1}{1-q} = p\frac{1}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

Variance:

$$\sum_{k=1}^{\infty} k^2 f(k) = p \sum_{k=1}^{\infty} k^2 q^{k-1} = p \sum_{k=1}^{\infty} \frac{d}{dq}(q\frac{d}{dq}q^k) = p\frac{d}{dq}(q\frac{d}{dq}\frac{1}{1-q}) = p\frac{d}{dq}(q\frac{1}{(1-q)^2}) = p\Bigg[\frac{1}{(1-q)^2} + \frac{2q}{(1-q)^3}$$

$$var(X) = \frac{1}{p} + \frac{2(1-p)}{p^2} - \frac{1}{p^2} = \frac{1}{p^2} - \frac{1}{p} = \frac{1-p}{p^2} = \frac{q}{p^2}$$

Negative Binomial Distribution

How many times does it take to get r 6's when rolling a die?