Math 191 Notes, 2003 November 20

Bivariate stuff continued

$$f_{X,Y}(x,y) = \begin{cases} 2 & (x,y) \text{ inside triangle} \\ 0 & \text{otherwise} \end{cases}$$

Choice 1: $U = X + Y, V = Y/X$.

$$J(u,v) = \frac{u}{(1+v)^2} \qquad f_{U,V}(u,v) = \frac{2u}{1+v^2}, \quad 0 \le u \le 1, 0 \le v$$

Choice 2: $U = X + Y, V = X - Y$

$$J(u,v) = -1/2 \quad f_{U,V}(u,v) = \begin{cases} 1 & 0 \le u \le 1, \quad -u \le v \le u \\ 0 & \end{cases}$$

What is $f_{U|V}(u,v)$. Choice 1:

$$f_{U|V}(u,v) = \frac{f_{U,V}(u,v)}{f_V(v)} = \frac{\frac{2u}{(1+v)^2}}{\frac{1}{(1+v)^2}} = 2u, \qquad f_{U|V}(u,1) = 2u$$

Choice 2[1]:

$$f_{U|V} = \frac{f_{U,V}(u,v)}{f_V(v)} = \frac{1}{1-|v|}, \qquad f_{U|V}(u,0) = 1$$

Geometric probability

$$f_X(x) = \lambda e^{-\lambda x}, \qquad x > 0$$
$$f_Y(y) = \lambda e^{-\lambda y}, \qquad y > 0$$
$$f_{X,Y}(x,y) = \begin{cases} \lambda^2 e^{-\lambda(x+y)} & x, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Poor choice: $U = X + Y, V = X - Y$

$$J(u,v) = -\frac{1}{2}$$

$$f_{U,V}(u,v) = \frac{1}{2}\lambda^2 e^{-\lambda u}, \qquad u \ge 0, -u \le v \le u$$

Now let's solve this problem "once and for all."

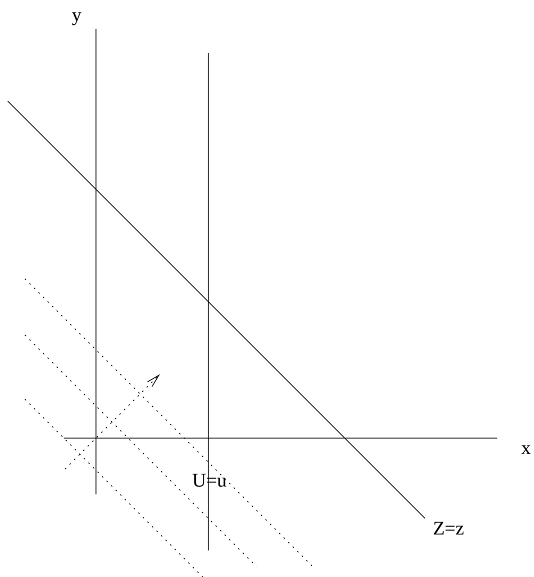
[1] $f_V(v) = \int_{-\infty}^{\infty} f_{U,V}(u,v)du = \int_{|v|}^{1} 1\, du = 1 - |v|$. The first expression is useless

Often we have we have two random variables whose sum we are interested in. First, let's look at the standard formula for the sum $Z = X + Y$. The standard formula is not unique.

$$Z = X + Y \qquad U = X$$

$$X = U \qquad Y = Z - U$$

$$J(u, z) = 1 \cdot 1 - 0 = 1$$



Safe way:

$$F_Z(z) = \mathbb{P}(X + Y \leq z) = \int_{-\infty}^{z} dv \int_{-\infty}^{\infty} du f_{X,Y}(u, v - u)$$

$$f_Z(z) = F'_Z(z) = \int_{-\infty}^{\infty} du f_{X,Y}(u, z - u)$$

$$\boxed{= \int_{-\infty}^{\infty} f_{X,Y}(x, z - x) dx}$$

Another way:

$$f_{U,Z}(u, z) = f_{X,Y}(u, z - u)$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(u, z - u) du$$

Yet Another Way: $Z = X + Y, V = X - Y$

$$X = \frac{Z + V}{2} \qquad Y = \frac{Z - V}{2}$$

$$J(z, v) = -1/2$$

$$f_{Z,V}(z,v) = \frac{1}{2} f_{X,Y}\left(\frac{z+v}{2}, \frac{z-v}{2}\right)$$

$$f_Z(z) = \frac{1}{2}\int_{-\infty}^{\infty} f_{X,Y}\left(\frac{z+v}{2}, \frac{z-v}{2}\right) dv$$

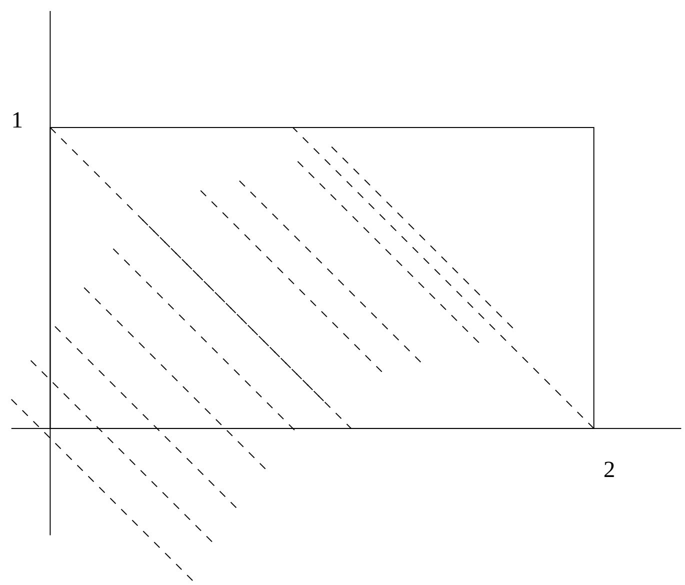
This is a change of variables of the original formula.

Important special case: X, Y are independent:

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

$$f_Z(z) = \frac{1}{2}\int_{-\infty}^{\infty} f_X\left(\frac{z+x}{2}\right) f_Y\left(\frac{z-x}{2}\right) dx$$

Look at an example. X is uniform on $[0,2]$. The random variable Y is uniform on $[0,1]$. We want the density function for the sum of X and Y. $Z = X + Y$. What is $f_Z(z)$?



$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2} & \text{in rectangle} \\ 0 & \text{outside} \end{cases}$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx$$

$$f_Z(z) = \int_0^z \frac{1}{2} f_Y(z-x) dx$$

3 cases:

- $z < 1$. $f_Z(z) = \int_0^z \frac{1}{2} dx = \frac{z}{2}$
- $1 \leq z \leq 2$. $f_Z(z) = \int_{z-1}^z \frac{1}{2} dx = \frac{1}{2}$

- $z > 2$. $\int_{z-1}^{2} \frac{1}{2} dx = \left[\frac{1}{2}x\right]_{z-1}^{2} = \frac{1}{2}(3-z)$

Independent Exponential distributions, with same λ

$$Z = X + Y$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx = \int \left(\lambda e^{-\lambda x}\right)\left(\lambda e^{-\lambda(z-x)}\right) dx$$

$$= \lambda^2 e^{-\lambda z} \int_0^z dx = \lambda^2 z e^{-\lambda z}$$

Independent normal, mean 0, variance 1

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

$$f_Z(z) = \frac{1}{2} \int_{-\infty}^{\infty} f_{X,Y}\left(\frac{z+v}{2}, \frac{z-v}{2}\right) dv$$

$$= \frac{1}{2\pi} \frac{1}{2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[\frac{z^2+2vz+v^2+z^2-2vz+v^2}{4}\right]} dv$$

$$= \frac{1}{4\pi} e^{-z^2/4} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\frac{v^2}{2}} dv = \frac{1}{2\sqrt{\pi}} e^{-z^2/4} \frac{1}{\sqrt{2\pi}\sqrt{2}} \int_{-\infty} \infty e^{-\frac{1}{2}\cdot\frac{v^2}{2}} dv$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-z^2/4} = \frac{1}{2\sqrt{\pi}} e^{-z^2/4}$$

There is also a change of variables that makes this work, but it is often harder to do than finding a second random variable.