Math 191 Notes, 2003 December 16

Random Walk via Generating Functions

Say we step right with probability p, left with probability $q = 1 - p$.

$$p_r(n) = \mathbb{P}(\text{in level } r \text{ after } n \text{ steps})$$
$$f_r(n) = \mathbb{P}(\text{be in or return to level } r \text{ for the first time after } n \text{ steps})$$

$f_r(n)$ is "defective" (not a probability distribution) because there is a drift and may sum to be less than 1.

$$F_0(s) = \sum_{n=1}^{\infty} f_0(n)s^n \qquad P_0(s) = \sum_{n=0}^{\infty} p_0(n)s^n$$

Condition on r, time of first return.

$$\begin{aligned}
p_0(n) &= \sum_{r=1}^{n} f_0(r)p_0(n-r) \\
\sum_{n=1}^{\infty} p_0(n)s^n &= \sum_{n=1}^{\infty}\sum_{r=1}^{n} f_0(r)p_0(n-r)s^n \\
&= \sum_{r=1}^{\infty} f_0(r)s^r \sum_{n=r}^{\infty} p_0(n-r)s^{n-r} \\
P_0(s) - 1 &= \left(\sum_{r=1}^{\infty} f_0(r)s^r\right)\left(\sum_{j=0}^{\infty} p_0(j)s^j\right) = F_0(s)P_0(s) \\
F_0(s) &= 1 - \frac{1}{P_0(s)}
\end{aligned}$$

Now,

$$p_0(n) = \binom{n}{n/2} p^{n/2} q^{n/2}$$

which is nonzero for even n.

$$P_0(s) = \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \binom{n}{n/2} p^{n/2} q^{n/2} s^n$$

Consider the following binomial expansion:

$$(1-4x)^{-1/2} = 1 + \frac{1}{2}(4x) + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}(4x)^2 + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}(4x)^3 + \cdots$$

Remarkably, there is an identity that gives us exactly what we want, the coefficients are actually $\binom{2}{1}, \binom{4}{2}, \binom{6}{3}, \ldots$. So then set $n = 2m$.

$$P_0(s) = \sum_{m=0}^{\infty} \binom{2m}{m} (pqs^2)^m$$

$$P_0(s) = (1 - 4pqs^2)^{-1/2}$$
$$F_0(s) = 1 - (1 - 4pqs^2)^{1/2}$$

That is the generating function for arriving at 0 for the first time after s steps. In particular,

$$F_0(1) = \sum_{n=1}^{\infty} f_0(n) = \mathbb{P}(\text{ever returning})$$

$$F_0(1) = 1 - \sqrt{1 - 4p + 4p^2} = 1 - |1 - 2p| = 1 - |q - p|$$

What is $f_r(n)$, the probability of first visiting level r after n steps?

$$f_r(n) = \sum_{k=1}^{n-r+1} f_1(k) f_{r-1}(n-k)$$

Looks like a convolution, good: define

$$F_1(s) = \sum_{n=1}^{\infty} f_1(n) s^n$$

Also

$$\sum_{n=r}^{\infty} f_r(n) s^n = \sum_{n=r}^{\infty} \sum_{k=1}^{n-r+1} f_1(k) f_{r-1}(n-k) s^n = \sum_{k=1}^{\infty} f_1(k) s^k \sum_{n=r+k-1}^{\infty} f_{r-1}(n-k) s^{n-k}$$
$$= \sum_{k=1}^{\infty} f_1(k) s^k \sum_{j=r-1}^{\infty} f_{r-1}(j) s^j = F_1(s) F_{r-1}(s)$$

We can see from this expression that the general expression

$$F_r(s) = [F_1(s)]^r$$

must be true. But this was obvious anyway; the sum of r random variables with $F_1(s)$ as the generating function.

$$f_1(1) = p$$
$$f_1(n) = q f_2(n-1) \qquad (n > 1)$$

$$\sum_{n=2}^{\infty} f_1(n)s^n = q\sum_{n=3}^{\infty} f_2(n-1)s^n$$

$$F_1(s) - ps = qs\sum_{n=3}^{\infty} f_2(n-1)s^{n-1} = qsF_2(s)$$

$$F_r(s) = [F_1(s)]^r$$

Note that $F_1(0) = f_1(0) = 0$.

$$F_1(s) - ps = qs(F_1(s))^2$$

$$qsF_1(s)^2 - F_1(s) + ps = 0$$

$$\boxed{F_1(s) = \frac{1 - \sqrt{1 - 4pqs^2}}{2qs}}$$

Now let's check this with an example:

$$F_1(1) = \frac{1 - \sqrt{1 - 4pq}}{2q} = \frac{1 - |p - q|}{2q}$$

If $p > q$,

$$F_1(1) = \frac{1 - (p - q)}{2q} = \frac{1 - p + q}{2q} = 1$$

If $p < q$,

$$F_1(1) = \frac{1 - (p - q)}{2q} = \frac{1 - q + p}{2q} = \frac{p}{q}$$

Check:

$$F_{-1}(s) = \frac{1 - \sqrt{1 - 4qps^2}}{2ps}$$

because we are interchanging p and q.

$$f_0(n) = pf_{-1}(n-1) + qf_1(n-1)$$

$$\sum_{n=1}^{\infty} f_0(n)s^n = \sum (ps)f_{-1}(n-1)s^{n-1} + \sum qsf_1(n-1)s^{n-1}$$

$$F_0(s) = psF_{-1}(s) + qsF_1(s)$$

Expand this out to get

$$F_0(s) = 1 - \sqrt{1 - 4pqs^2}$$

and everthing checks.

Tied-down leads for random walk

Say Harvard and Stanford play $2n$ games, series is tied:

$$\mathbb{P}(\text{Harvard is ahead for } 2k \text{ intervals}) = \frac{1}{n+1}$$

for $k = 0, \dots, n$.