MATHEMATICS 191, FALL 2003
MATHEMATICAL PROBABILITY
Outline #5 (Discrete Random Variables and Expectation)

Reference: G&S, Sections 3.1 through 3.7

1. Write down the probability mass functions for the binomial and Poisson distributions, and show that the associated distribution functions meet the requirements of a discrete distribution function. Show how to convert any convergent infinite series of positive terms into a mass function.

2. Define what it means for discrete random variables to be independent. Now suppose that a coin is tossed N times, where the random variable N has the Poisson distribution. The number of heads X and the number of tails Y are also random variables. Show that they are independent.

3. Define the expectation of a random variable X. Show that in the carnival game of "Chuck-A-Luck" (outline 2) the expected number of occurrences of the chosen number is $\frac{1}{2}$.

4. State and prove the "law of the unconscious statistician." Give an example to show that using this law can lead to different arithmetic than strict application of the definition of expectation. Then use it to show that expectation is a linear function on the space of random variables:

$$\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$$

.

5. Define variance, and prove that var(X)=$\mathbb{E}(X^2) - \mathbb{E}(X)^2$

6. Define what it means for random variables to be uncorrelated. Prove that independent random variables are uncorrelated, and give an example to show that the converse is not necessarily true. Prove that for uncorrelated random variables X and Y, $\text{var}(X+Y) = \text{var}(X)+\text{var}(Y)$, but show that variance is not a linear function on the space of random variables.

7. Give an example of a random variable whose expectation is undefined because the infinite series that defines it is only conditionally convergent, not absolutely convergent. Also give an example of a random variable whose expectation is infinite but for which a reasonable person would pay only a finite amount to play a game whose payoff is given by this random variable.

8. Using indicator functions, prove that if n letters are placed at random into n matching envelopes, the probability that no letter ends up in the correct envelope approaches e^{-1} as $n \to \infty$.

9. Explain the application of the "probabilistic method" that is described on page 59 of G&S, and show that it is really just an example of counting something in two alternative ways. Give a simpler example (four letters in three mailboxes) that illustrates more clearly the connection with the pigeonhole principle.

10. Show how the Poisson distribution can be obtained as a limiting case of the binomial distribution, and derive the expectation and variance for the Poisson distribution by this limiting process.

11. Describe a process that leads to a random variable with a geometric distribution and a generalization of this process that leads to a negative binomial distribution. Calculate the expectation and variance for the negative binomial distribution by treating it as a sum of independent random variables.

12. Prove that two random variables are independent if and only if their joint mass function $f(x, y)$ can be expressed as a product $g(x)h(y)$. Show how this fact explains the independence of X and Y in item 2 above.

13. Define the correlation $\rho(X, Y)$ of two random variables. By proving the Cauchy-Schwarz inequality, show that $|\rho(X, Y)| \leq 1$, with equality only if there is a linear relationship between X and Y.

14. For two random variables X and Y, define the conditional distribution function, conditional mass function, and conditional expectation of Y given $X = x$. Prove that $\mathbb{E}(\mathbb{E}(Y|X)) = \mathbb{E}(Y)$

15. (This is the same as the example on p. 68 of G&S, with a different story line)
    High-school seniors apply to N colleges, where N has the Poisson distribution with parameter λ. Each applicant is admitted by each college independently with probability p. Calculate the following:

    (a) the mean number of colleges to which students applied.

    (b) the mean number of acceptances received by students who applied to N colleges.

    (c) the mean number of acceptances per student.

    (d) the mean number of applications filed by students who were accepted at K colleges.