MATHEMATICS 191, FALL 2003
MATHEMATICAL PROBABILITY
Outline #4 (Random Variables)

Reference: G&S, Chapter 2

1. Define "random variable," paying attention to the restriction in the definition that is imposed by the choice of σ-field. Give a couple of examples of random variables that can be associated with flipping coins and rolling dice.

2. Define the distribution function of a random variable, and prove the three properties listed as Lemma 6 on p. 28 of G&S. Make it clear why the use of $<$ and $\leq$ is correct in (b) and why $F(x)$ need only be right-continuous. Give an example of a discontinuous distribution function.

3. Describe the Bernoulli variable associated with a flip of a coin (not necessarily a fair coin), and sketch a graph of its distribution function. Explain how this is a special case of an indicator function. Invent an example to illustrate identity (10) at the bottom of p. 29.

4. State the "law of averages" and prove it by deriving Bernstein's identity.

5. Explain what is meant by a discrete random variable and by a continuous random variable. Give one example of a random variable that is neither discrete nor continuous because it is a mixture. Show that the "Schwarzkopf attack" in outline 1 led to a singular random variable that was neither discrete nor continuous nor a mixture.

6. Prove that the set of points x in a sample space Ω for which $\mathbb{P}(x) > 0$ is countable, even if Ω is uncountable.

   This proof is not in G&S, but it is easy. Just consider, for each positive n, the set of points for which

$$\frac{1}{n+1} < \mathbb{P}(x) \leq \frac{1}{n}$$

   Each such set is finite, so you have a countable union of finite sets.

7. Give examples of random variables that can be associated with a dart board.

8. Define the joint distribution function of a random vector. Give an example of a continuous random vector and of a discrete random vector.