Math 191 Notes, 2004 January 8

Weak Law of Large Numbers

Special Case

Imagine lottery tickets:

$$\mathbb{P}(X = 2) = 0.5, \qquad \mathbb{P}(X = 0) = 0.5$$

Generating function:

$$G_X(s) = \frac{1}{2}(1 + s^2)$$

$$\text{For } \frac{X}{n}, G_{X/n}(s) = \frac{1}{2}(1 + s^{2/n})$$

For n copies of X/n: $S_n = \sum_{n=1}^{\infty} X_n$, look at S_n/n (average).

$$G_n(s) = \left[\frac{1}{2}(1 + s^{2/n})\right]^n$$

Now we must rewrite this if we are going to take limits. Let

$$s^{2/n} = e^{\frac{2 \log s}{n}} \approx \left(1 + \frac{2 \log s}{n} + \cdots\right)$$

So now,

$$\begin{aligned} \lim_{n \to \infty} G_n(s) &= \lim_{n \to \infty} \left(\frac{1}{2}\left(1 + 1 + \frac{2 \log s}{n} + \cdots\right)\right)^n \\ &= \lim_{n \to \infty} \left(1 + \frac{\log s}{n} + \cdots\right)^n = e^{\log s} = s \end{aligned}$$

Central Limit Theorem

$$\mathbb{P}(X = 1) = 0.5, \qquad \mathbb{P}(x = -1) = 0.5$$

The mean of this is 0, the variance is 1.

$$G_X(s) = \frac{1}{2}(s^{-1} + s)$$

"All we want is generating functions where the series converges in some neighborhood of 1." The derivatives at 1 are all that are important to us because the expectation and variance are all related to the derivatives at 1.

$$\begin{aligned}
G_{X/\sqrt{n}}(s) &= \frac{1}{2}(s^{-1/\sqrt{n}} + s^{1/\sqrt{n}}) = \frac{1}{2}\left(e^{-\frac{\log s}{\sqrt{n}}} + e^{\frac{\log s}{\sqrt{n}}}\right) \\
&\approx \frac{1}{2}\left(1 - \frac{\log s}{n} + \frac{1}{2}\cdot\frac{\log^2 s}{n} + \cdots + 1 + \frac{\log s}{n} + \frac{1}{2}\cdot\frac{\log^2 s}{n}\right) \\
&= \left(1 + \frac{1}{2}\frac{\log^2 s}{n} + \cdots\right)
\end{aligned}$$

For n copies,

$$G_n(s) = (G_{X/n}(s))^n = \left(1 + \frac{1}{2}\frac{\log^2 s}{n}\right)^n \to e^{\frac{\log^2 s}{n}}$$

Now,

$$N(0,1) : f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

$$\begin{aligned}
G_{\text{normal}}(s) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} s^x e^{-x^2/2}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{x\log s}e^{-x^2/2}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{(\overbrace{x-\log s}^{u})^2}{2}} e^{\frac{\log^2 s}{2}}dx = e^{\frac{\log^2 s}{2}}
\end{aligned}$$

Weak Law of Large Numbers: General Case

Assume X has expectation $\mathbb{E}(X) = \mu$.

$$G_X(1) = 1, G'(1) = \mu$$

$$G_X(s) - G_X(1) = G_X'(1)(s-1) + O(s-1)$$

$$\begin{aligned}
G_X(s) &= 1 + G_X'(1)(s-1) + O(s-1) \\
&= 1 + \mu(s-1) + O(s-1) \\
G_{X/n}(s) &= 1 + \mu(s^{1/n} - 1) + \cdots = 1 + \mu\left(e^{\frac{\log sn}{-}}1\right) + \cdots \\
&= 1 + \mu\left(1 + \frac{\log s}{n} - 1 + \cdots\right) + \cdots \\
&= 1 + \frac{\mu \log s}{n} \text{ as } n \to \infty \lim_{n\to\infty} \\
G_n(s) &= \lim_{n\to\infty}\left(G_{X/n}(s)\right)^n = \lim_{n\to\infty}\left(1 + \frac{\mu\log s}{n}\right)^n = ???
\end{aligned}$$

Central Limit Theorem: General Case

Choose X with mean $\mathbb{E}(X) = 0$, variance σ^2. Then $\mathbb{E}(X^2) = \sigma^2$.

$$\begin{aligned}
G_X(s) &= G(1) + G'(1)(s-1) + \frac{1}{2}G''(1)(s-1)^2 + O(s-1)^2 \\
&= 1 + 0 + \frac{1}{2}\sigma^2(s-1)^2 + \cdots \\
G_{\frac{X}{\sqrt{n\sigma^2}}}(s) &= 1 + \frac{1}{2}\sigma^2\left(s^{\frac{1}{\sqrt{n\sigma^2}}} - 1\right)^2 + \cdots \\
&= 1 + \frac{1}{2}\sigma^2\left(e^{\frac{\log s}{\sqrt{n\sigma^2}}-1}\right)^2 + \cdots \\
&= 1 + \frac{1}{2}\sigma^2\left(\frac{\log s}{\sqrt{n\sigma^2}}\right)^2 + \cdots \\
&= 1 + \frac{1}{2}\sigma^2\frac{\log^2 s}{n\sigma^2}
\end{aligned}$$

Add n copies: $\frac{1}{\sqrt{n}}S_n$.

$$\begin{aligned}
G_n(s) = \left(G_{\frac{X}{n}}(s)\right)^n &= \left(1 + \frac{\log^2 s}{2n}\right)^n \\
\lim_{n\to\infty} G_n(s) = \lim_{n\to\infty}\left(1 + \frac{\log^2 s}{2n}\right)^n &= e^{\frac{\log^2 s}{2}}
\end{aligned}$$

This is the same as the normal distribution.

Markov's Inequality (General Case)

Suppose you have some function $h(x)$ that assumes only non-negative values. For example $|x|, x^2, \ldots$. And suppose we have a random variable X. Choose some positive c.

$$\begin{aligned}
\mathbb{E}(h(X)) &= \mathbb{P}(h(x) > c)\mathbb{E}(h(x)|h(x) > c) + \mathbb{P}(h(x) \leq c)\mathbb{E}(h(x)|h(x) \leq c) \\
\mathbb{E}(h(X)) &\geq \mathbb{P}(h(x) > c) \cdot c \\
\mathbb{P}(h(X) > c) &\leq \frac{\mathbb{E}(h(x))}{c}
\end{aligned}$$

Chebychoev's Inequalities

$$\mathbb{P}(|X| > a) = \mathbb{P}(X^2 > a^2) \leq \frac{\mathbb{E}(X^2)}{a^2}$$

If $\mathbb{E}(X) = 0$,

$$\mathbb{P}(|X| > a) \leq \frac{\text{var}X}{a^2}$$

. Now,

$$S_n = \sum_{i=1}^{n} X_n, \qquad \text{var}(S_n) = n \ \text{var}(X)$$

Assume $\mathbb{E}(X) = 0$.

$$\begin{aligned}\mathbb{P}\left(\left|\frac{S_n}{n}\right| > \epsilon\right) &= \mathbb{P}(|S_n| > n\epsilon) \\ &\leq \frac{\text{var } S_n}{n^2\epsilon^2} = \frac{n \text{ var } X}{n^2\epsilon^2} = \frac{\text{var } X}{n\epsilon^2}\end{aligned}$$

This weak law of large numbers is not satisfying because the probability goes down by a factor linear with respect to the number of independent variables. Unfortunately, that means that the absolute number is the same problem.

Strong Law of Large Numbers

Assume $\mathbb{E}(X) = 0, \mathbb{E}(X^2) = \sigma^2, \mathbb{E}(X^4) = M$. Last condition is not necessary but makes this proof easier.

$$\begin{aligned}S_n &= X_1 + X_2 + \cdots + X_n \\ S_n^4 &= (X_1 + X_2 + \cdots + X_n)^4\end{aligned}$$

Consider $\mathbb{E}(S_n^4)$. The second line as n^4 terms. 5 cases.

- $\mathbb{E}(X_1^4)$: n such terms.
- $\mathbb{E}(X_1^3 X_2) = \mathbb{E}(X_1^3)\mathbb{E}(X_2) = 0$
- $\mathbb{E}(X_1^2 X_2^2) = \mathbb{E}(X_1^2)\mathbb{E}(X_2^2) = (\text{var } X)^2 = \sigma^4$. $6\binom{n}{2} = 3n(n-1)$ terms.
- $\mathbb{E}(X_1^2 X_2 X_3) = 0$
- $\mathbb{E}(X_1 X_2 X_3 X_4) = 0$.

We want to know:

$$\begin{aligned}\mathbb{P}(|S_n| > n\epsilon) &= \mathbb{P}(|S_n| > n\epsilon) \\ &= \mathbb{P}(S_n^4 > n^4\epsilon^4) \leq \frac{\mathbb{E}(S_n^4)}{n^4\epsilon^4} \\ \mathbb{P}(|S_n/n| > \epsilon) &\leq \frac{nM + 3n(n-1)\sigma^4}{n^4\epsilon\epsilon^4} \approx \frac{C}{n^2\epsilon^4}\end{aligned}$$

And this converges. So this proves that exceeding the average cannot occur infinitely often!

Let's pick some N. What's the probability of:

$$\mathbb{P}\left(\bigcup_{n=N}^{\infty}\left(\left|\frac{S_n}{n}\right| > \epsilon\right)\right) \leq \sum_{n=N}^{\infty} \mathbb{P}(|S_n/n| \geq \epsilon)$$
$$\leq \sum_{n=N}^{\infty} \frac{C}{n^2\epsilon^4}$$

By choosing N as large as we like, we can make the RHS arbitrarily small (because this is a convergent sequence).