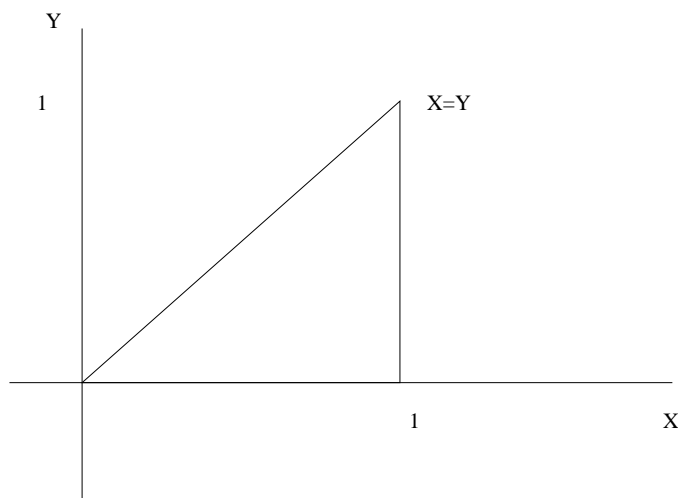


Math 191 Notes, 2003 December 2

Continuation of example last time

X, Y independent, uniform on $[0, l]$. What is $\mathbb{E}(|X - Y|)$?

1. Straightforward method: Let $0 \leq Y \leq X \leq 1$ and compute $\mathbb{E}(X - Y)$. Not trivial, because now X and Y are no longer independent!



$$f_{X,Y}(x, y) = 2$$

$$\begin{aligned} \int_0^1 2dx \int_0^y dy(x - y) &= \int_0^1 2xdx \int_0^x dy - \int_0^1 2dx \int_0^x ydy \\ &= \int_0^1 2x^2dx - \int_0^1 x^2dx = \int_0^1 x^2dx = \frac{1}{3} \end{aligned}$$

2. The answer we want should be linear in l (make sure you understand this!). $\mathbb{E}(|X - Y|) = \alpha l$. Divide the segment into two halves: the probability that they happen on different halves is $\frac{1}{2}$, as is the probability of them being in the same half.

$$\alpha l = \underbrace{\frac{1}{2} \cdot \frac{1}{2} l}_{\text{same half}} + \underbrace{\frac{1}{2} \cdot \frac{1}{2} \alpha l}_{\text{opposite halves}}$$

$$\frac{3}{4} \alpha l = \frac{1}{4} l$$

$$\alpha = \frac{1}{3}$$

3. Crofton method: turning a continuous probability problem into a differential equation and solving that. Imagine that the integral runs from 0 to z , and imagine what would be different if it ran from 0 to $z + h$. Here is the clever idea:



$$\mathbb{P}(X \in [0, z]) = \frac{z}{z+h}$$

$$\begin{aligned} \mathbb{P}(X, Y \in [0, z]) &= \left(\frac{z}{z+h}\right)^2 \\ &= \frac{1}{\left(1 + \frac{h}{z}\right)^2} \approx 1 - \frac{2h}{z} + \dots \end{aligned}$$

$$\mathbb{E}(|X - Y|) = L(z)$$

$$L(z+h) = \left(1 - \frac{2h}{z}\right) L(z) + \frac{2h}{z} \cdot \frac{z+h}{2}$$

$$L(z+h) - L(z) + \frac{2h}{z} L(z) = \frac{2h}{z} \cdot \frac{z}{2}$$

$$\frac{L(z+h) - L(z)}{h} + \frac{2L(z)}{z} = 1$$

Note that everything here is to first-order in h , so we throw away any h^2 terms. Now we take the limit as $h \rightarrow 0$,

$$\frac{dL}{dz} + \frac{2L}{z} = 1$$

Separation of variables is not directly possible, but don't despair! Introduce some integrating factor $\mu(z)$ that will get rid of the annoying second term:

$$\begin{aligned} \mu \frac{dL}{dz} + \mu \frac{2L}{z} &= \mu \\ \frac{d}{dz}(\mu L) - L \frac{d\mu}{dz} + \mu \frac{2L}{z} &= \mu \end{aligned}$$

Now, we don't want the last two terms on the left-hand side, so set their values equal to each other...

$$L \frac{d\mu}{dz} = \mu \frac{2L}{z}$$

$$\frac{d\mu}{\mu} = \frac{2dz}{z}$$

$$\log \mu = 2 \log z$$

$$\mu = z^2$$

Now, let's go back to our original equation and plug it in:

$$z^2 \left(\frac{dL}{dz} + \frac{2L}{z} \right) = z^2$$

$$\frac{d}{dz} (Lz^2) = z^2$$

$$Lz^2 = \frac{z^3}{3} + C$$

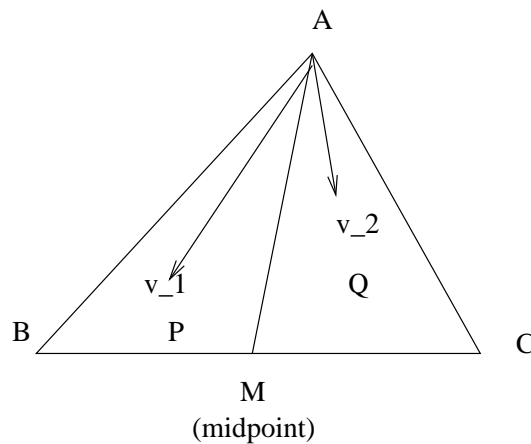
Boundary condition gives us $z = 0 \Rightarrow L(z) = 0$, so $C = 0$, so

$$L = \frac{z}{3}$$

The quiz will have something like this, using Crofton's method.

Triangle problem

Choose random point \vec{v}_1 going to point P , in $\triangle ABM$, \vec{v}_2 going to Q in $\triangle AMC$.

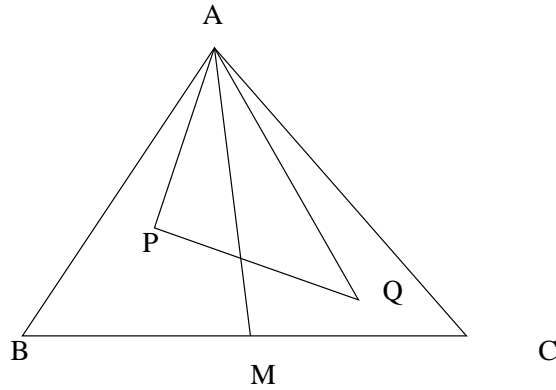


$$\begin{aligned}
\text{Area of } |APQ| &= \frac{1}{2}|\vec{v}_1 \times \vec{v}_2| \\
&= \frac{1}{2}|x_1y_2 - x_2y_1| = \frac{1}{2}(x_1y_2 - x_2y_1) \\
\mathbb{E}(\text{Area}) &= \frac{1}{2}(\mathbb{E}(x_1)\mathbb{E}(y_2) - \mathbb{E}(x_2)\mathbb{E}(y_1))
\end{aligned}$$

The expected value for the location of a point inside the triangle is at the center of mass of the triangle. Let the center of mass of $\triangle ABM$ be \vec{G}_1 , and of $\triangle AMC$ to be \vec{G}_2 :

$$\begin{aligned}
&= \frac{1}{2}|\vec{G}_1 \times \vec{G}_2| = \text{Area}(AG_1G_2) \\
&= \frac{1}{2}(\text{base})(\text{altitude}) = \frac{1}{2} \cdot \frac{1}{3}b \cdot \frac{2}{3}h \\
&= \frac{2}{9} \left(\frac{1}{2}bh \right) = \frac{2}{9}|ABC|
\end{aligned}$$

Now, let's do the original problem of picking two random points anywhere in the triangle.

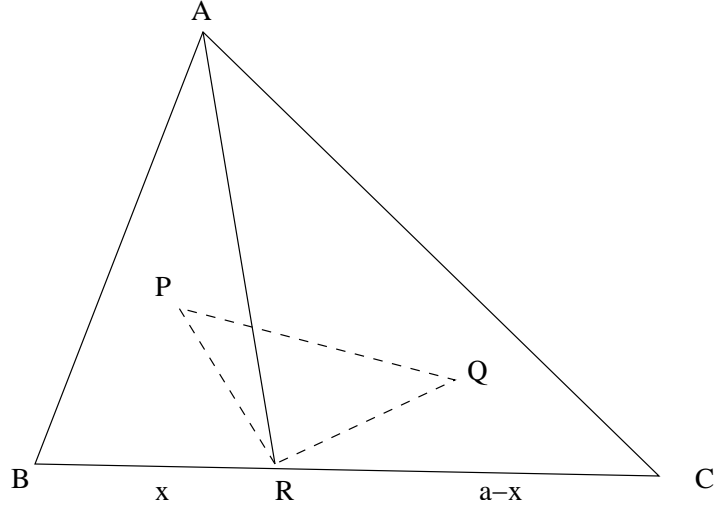


$$\mathbb{E}(|APQ|) = \alpha|ABC|$$

The probability that P, Q are in different halves or in the same half are both $p = \frac{1}{2}$.

$$\begin{aligned}
\alpha|ABC| &= \frac{1}{2} \cdot \frac{1}{2}\alpha|ABC| + \frac{1}{2} \cdot \frac{2}{9}|ABC| \\
\alpha &= \frac{4}{27}
\end{aligned}$$

Another method: Crofton's method. Lemma with 3 Cases:



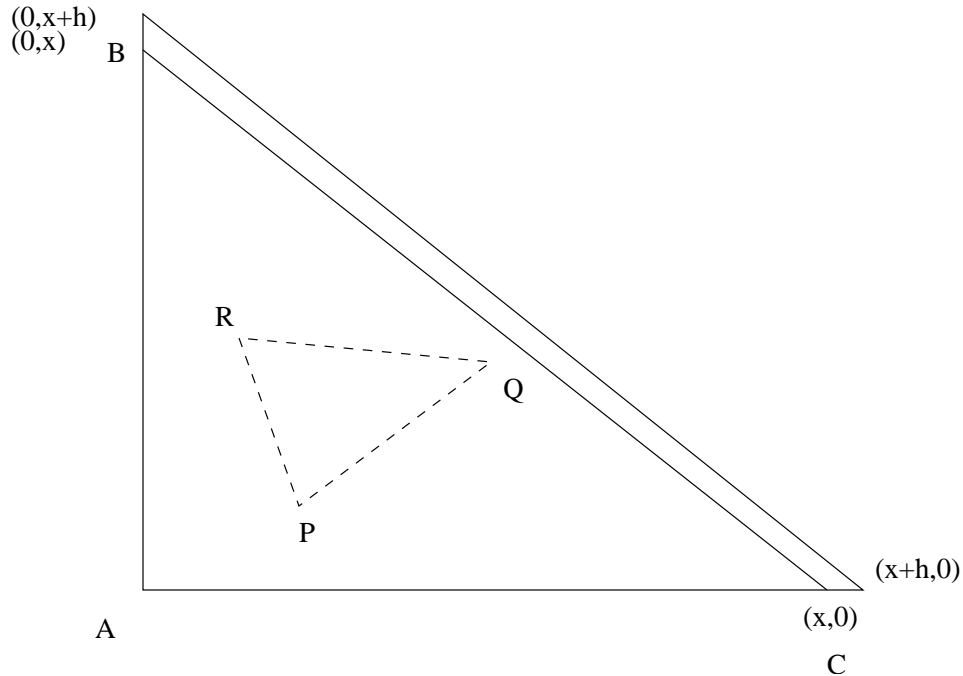
1. P, Q both in left triangle, $p = (x/a)^2$
2. P, Q both in right triangle, $p = \left(\frac{a-x}{x}\right)^2$
3. P, Q in different halves $p = \frac{2x(a-x)}{a^2}$

$$\mathbb{E}(|RPQ|) = \mathbb{E} \left(\underbrace{\frac{x^2}{a} \cdot \frac{4}{27} \frac{x}{a} |ABC|}_{\#1} + \underbrace{\left(\frac{a-x}{a}\right)^2 \cdot \frac{4}{27} \frac{a-x}{a} |ABC|}_{\#2} + \underbrace{\frac{2x(a-x)}{a^2} \frac{1}{2} \cdot \frac{2}{9} |ABC|}_{\#3} \right)$$

where the extra factor of $1/2$ in the last term is because we are using the bottom as the altitude instead of the top.

$$\mathbb{E}(|RPQ|) = \frac{1}{27}|ABC| + \frac{1}{27}|ABC| + \frac{1}{27}|ABC| = \frac{1}{9}|ABC|$$

where we calculated each of the integrals individually. Now we're going to choose a convenient triangle to finish this off.



$$\mathbb{P}(P \in \triangle ABC) = \left(\frac{x}{x+h}\right)^2$$

$$\mathbb{P}(P, Q, R \in \triangle ABC) = \left(\frac{x}{x+h}\right)^6 = \frac{1}{\left(1 + \frac{h}{x}\right)^6} = 1 - \frac{6h}{x} + \dots$$

Now use Crofton's method. Let $A(x)$ be the expected area of $\triangle PQR$ in isosceles right triangle of side x .

$$A(x+h) = \underbrace{\left(1 - \frac{6h}{x}\right)}_{\mathbb{P}(P, Q, R \in |ABC|)} A(x) + \underbrace{\frac{6h}{x} \cdot \frac{1}{9} \cdot \frac{1}{2} x^2}_{\mathbb{P}(P, Q, \text{ or } R \in \text{edge})}$$

$$A(x+h) - A(x) + \frac{6h}{x} A(x) = \frac{1}{3} hx$$

$$\frac{A(x+h) - A(x)}{h} + \frac{6A}{x} = \frac{1}{3} x$$

Take limit:

$$\frac{dA}{dx} + \frac{6A}{x} = \frac{1}{3} x$$

Integrating factor is x^6

$$\begin{aligned}x^6 \frac{dA}{dx} + 6Ax^5 &= \frac{1}{3}x^7 \\ \frac{d}{dx} (x^6 A) &= \frac{1}{3}x^7 \\ x^6 A &= \frac{1}{24}x^8 + C\end{aligned}$$

$A(0) = 0$, so $C = 0$

$$A = \frac{1}{24}x^6 = \frac{1}{12}x^2 = \frac{1}{12}|ABC|$$

Thus by the equivalence of these properties under affine transformations, for *any* triangle ABC (not necessarily isosceles right), we have that

$$\mathbb{E}(|PQR|) = \frac{1}{12}|ABC|.$$

Done. Last comment, figure 4.1 in the book. At least try to find the answer when one of the points is out on the edge of the circle using Crofton's method.