Math 191 Notes, 2003 December 11

Problem of the points by generating functions

Ball: prob p, Strike: prob $q = 1 - p$. Midget must get m balls before n strikes: call this probability $p_{m,n}$.

$$p_{0,0} = 0$$

by convention. Also,

$$p_{m,0} = 0 \qquad p_{0,n} = 1$$

$$p_{m,n} = pp_{m-1,n} + qp_{m,n-1} \qquad m, n \geq 1$$

Define

$$G(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p_{m,n} x^m y^n$$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} p_{m,n} x^m y^n = p \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} p_{m-1,n} x^m y^n + q \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} p_{m,n-1} x^m y^n$$

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p_{m,n} x^m y^n - \sum_{n=0}^{\infty} p_{0,n} x^0 y^n = px \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} p_{m-1,n} x^{m-1} y^n + qy \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} p_{m,n-1} x^m y^{n-1}$$

Note that there is another term on the left but all its coefficients are zero so we are ignoring it. Now skip some steps (because we are low on time):

$$G(x, y) - \frac{y}{1-y} = px\, G(x, y) + qy \left[G(x, y) - \frac{y}{1-y} \right]$$

$$G(x, y)[1 - px - qy] = (1 - qy) \frac{y}{1-y}$$

$$G(x, y) = \frac{y}{1-y} \cdot \frac{1 - qy}{1 - px - qy}$$

And thus we have a closed-form solution to this problem. Now, of course, expanding this in a power series (if you do it badly) could be harder than solving the original problem. So rewrite as:

$$G(x, y) = \frac{y}{1-y} \left(1 - \frac{px}{1 - qy} \right)^{-1}$$

And then we can expand this pretty nicely.