MATHEMATICS 191, FALL 2004
MATHEMATICAL PROBABILITY
Assignment #7

Problems to be discussed in section on Monday, Nov. 22: Note: this is after the quiz, during the week of Thanksgiving. Sections will NOT meet on Monday, Nov. 15, but the CAs will hold review sessions(times and places TBA) on Wednesday the 17th for the quiz on the 18th.

All problems are from Grimmett and Stirzaker, 1000 Exercises in Probability. The solutions are all in the book!

Add together the number of letters in your first and last name. If the sum is odd, prepare problems 1,3, and 5. If it is even, prepare 2, 4, and 6.

1. Section 4.14, problem 2 (normalizing a density function).
2. Section 4.1 , problem 1b (normalizing another density function).
3. Section 4.3 , problem 3 (expectation of X^r).
4. Section 4.4, problem 3 (moments of the Cauchy distribution).
5. Section 4.3, problem 4 (relation among mean, median, and variance).
6. Section 4.4, problem 1 (gamma function for half-integers).

Problems to be handed in on Tuesday, November 16:

1. It was proved in lecture that the number of paths $N_{2n}(0,0) = \binom{2n}{n}$ that return to the origin after $2n$ steps is equal to the number of paths that start at level 0 and never return to the origin during $2n$ steps and is also equal to the number of paths that never go below level 0 during $2n$ steps. For these other two cases, an odd number of steps is also possible.

   (a) Show that the the number of paths that start at level 0 and never return to the origin during $2n+1$ steps is $2N_{2n}(0,0)$ if $n > 1$. Confirm this result by considering a 3-vote election and a 5-vote election involving B and K and listing the cases where there is never a tie.

   (b) Show that the the number of paths that start at level 0 and never go below level 0 during $2n+1$ steps is $\frac{1}{2}N_{2n+2}(0,0)$ if $n > 1$. Confirm this result by considering a 3-vote election and a 5-vote election involving B and K and listing the cases where B is never behind.

   (c) Consider a symmetric random walk of $2n$ steps that achieves its maximum level $M_{2n} > 0$ for the first time after i steps. Show that, for $k > 0$, the probability that this maximum is achieved for the first time after i steps is the same for $i = 2k+1$ as for $i = 2k$. (This is a followup to problem 3.11.28, discussed in section)

2. Verify the three discrete arc sine laws for the case $2n = 6, 2k = 2$ by considering the case of an election where 6 voters in turn cast ballots for B or K, each with probability $\frac{1}{2}$ for each candidate. There are 64 possible sequences of votes, and you should list all of them for which

   (a) (last visit to the origin)the last tie occurred with two votes left to be cast.

   (b) (sojourn times) B was ahead during precisely two of the six segments.

   (c) (maxima – even i)B had his maximum lead for the first time after two ballots had been counted.

   (d) (maxima– odd i)B had his maximum lead for the first time after three ballots had been counted.

   In the first two cases you should find $u_2 u_4$ sequences. If the third there are only half as many.

3. Do a geometric proof of the theorem that for for a random walk of $2n$ steps starting at 0, the number of paths that end at 0 is equal to the number of paths that never fall below 0 during the first $2n$ steps. Here is the basic idea, attributed by W. Feller to E. Nelson.

Consider a path that starts and ends in level 0. Let the first occurrence of the (global) minimum of this path be in level $-m$ after k steps. Reflect the portion of the path from 0 to k in the vertical line for time k, then slide it $2n - k$ units to the right and m levels up so that you can attach it to the right of $2n$. Show that this creates a path of $2n$ steps (with a different origin at time k, level $-m$) that never falls below its starting level. Explain how to invert the construction to establish a bijection between the two types of paths.

For the case where Harvard and Yale play 4 games, there are 6 sequences (like YHHY) that lead to a tied series after 4 games and 6 sequences (like HYHH) that have the property that Harvard is never behind in the series. Exhibit the one-to-one correspondence established by your construction.

4. Let f_{2n} denote the probability that a symmetric random walk returns to the origin (level 0) for the first time after $2n$ steps. Let u_{2n} denote the probability that a symmetric random walk returns to the origin (level 0) after $2n$ steps. As was proved in class, u_{2n} is also equal to the probability of a variety of other events.

   (a) By applying the ballot theorem to all the steps after the first, derive a simple formula for f_{2n} in terms of u_{2n} divided by an appropriate function of n. Check it for the cases $2n = 2, 4, 6$ by calculating f_{2n} for the formula, then listing or graphing all the paths that return to the origin (level 0) for the first time after $2n$ steps.

   (b) By noting that if a symmetric random walk returns to level 0 for the first time after $2n$ steps, the last step must be opposite to the first step, and then combining this observation with the result for the "impoverished student" problem from the last problem set, derive a formula for f_{2n} in terms of u_{2n-2} divided by an appropriate function of n. Show that this agrees with your previous formula.

5. Suppose that random variable X has the distribution function $F(x) = \frac{2}{\pi} \arcsin \sqrt{x}$ for $0 \le x \le 1$. Calculate its expectation (obvious by symmetry, but do the integral) and its variance.

6. X is a random variable that is uniformly distributed on $[0, \frac{\pi}{2}]$. For each of the following random variables, determine the distribution function and the density function, and sketch a graph of each.

   (a) $Y = \sin^2 X$.

   (b) $Z = \tan \frac{X}{2}$.

7. This optional question is an expanded version of one from last year's second quiz. You should be prepared for another one like it on Thursday! You may hand it in, but it will not count in the grading.

   Suppose that X is a random variable with a uniform distribution in $[0, 4]$ and that $Y = X^{\frac{3}{2}}$. So Y can assume any value in $[0, 8]$.

   (a) Find the distribution function $F_Y(y)$ for Y and find a density function $f_Y(y)$ for Y. It is sufficient to give formulas that are valid for $0 \le y \le 8$. Sketch graphs of the two functions.

   (b) Calculate the expectation of Y. There is more than one way to do this. Only one was required on the quiz, but for homework you should try all three, namely

   - the definition of expectation, integrating over y.
   - the law of the unconscious statistician, integrating over x.
   - the formula $\mathbb{E}(Y) = \int_0^\infty (1 - F_Y(y))dy$