MATHEMATICS 191, FALL 2004
MATHEMATICAL PROBABILITY
Outline #10 (Generating Functions)

Reference: PRP, Sections 5.1 through 5.3

1. Define the probability generating function $G_X(s)$ for a random variable X that takes on only non-negative integer values. Show that the product of two such generating functions is the generating function for their sum and that the nth power of $G_X(s)$ is the generating function for the sum of n independent copies of X.

2. As a trivial example of generating functions, let X assume values 1 and 2 with equal probability and let Y assume values 1, 2, and 3 with equal probability. Construct the generating function for $X + Y$ and confirm that it gives the right mass function for $X + Y$.

3. As a less trivial example of generating functions, let X, Y, and Z be three independent rolls of a standard fair die. Construct the generating function for $X + Y + Z$. By expanding it in an infinite series, determine the probability of rolling a total of 9 with 3 dice.

4. As yet another example of generating functions, construct the generating function for the sum of the rolls on two fair dice. By factoring this function differently, show that the same random variable can be achieved by summing the rolls on two suitably constructed "loaded" dice.

5. For a generating function $G(s)$, show the interpretation of $G(1)$, $G'(1)$, and $G''(1)$ in terms of expectations. Then show that the "moment generating function $M(t) = G(e^t)$" is even more useful for calculating expectations of powers of X by differentiation.

6. Construct the generating functions for the following discrete distributions:

   - the sum of n independent Bernoulli variables, each with probability p for "success."
   - a geometric distribution with probability p for "success," and the negative binomial distribution that is the sum of n of these.
   - the sum of two independent Poisson random variables, with parameters λ_1 and λ_2 respectively.

7. As an example of converting a recurrence into a generating function, revisit the "problem of the points." As a concrete example, think of a baseball player who stands at the plate taking pitch after pitch. The probability of a ball is p, of a strike is q, and $p_{m,n}$ denotes the probability of getting m balls before n strikes. Clearly, for $m, n \geq 1$, if we condition on whether the first pitch is a ball or a strike,

$$p_{m,n} = pp_{m-1,n} + qp_{m,n-1}$$

.

Let

$$G(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p_{m,n} x^m y^n.$$

Show that

$$G(x, y) = \frac{y}{1-y}(1 - \frac{px}{1-qy})^{-1}$$

8. As an example of converting a recurrence into a differential equation for a generating function, revisit the "matching problem ." Let p_n denote the probability that when n letters are stuffed randomly into n envelopes, no letter is in its correct envelope. Show that for $n \geq 3$,

$$np_n = p_{n-2} + (n-1)p_{n-1}$$

.

Let $G(s) = \sum_{n=1}^{\infty} p_n s^n$.

Find and solve a differential equation for $G(s)$ to show that

$$G(s) = \frac{e^{-s}}{1-s} - 1$$

9. If time permits, analyze the simple random walk by means of generating functions.