MATHEMATICS 191, FALL 2004
MATHEMATICAL PROBABILITY
Outline #4 (Random Variables)

Last modified: October 6, 2004
References: Because EP does everything with discrete random variables before introducing continuous random variables, this material is scattered throughout the book. PRP does an overview in chapter 2, leaving most details to later chapters. This outline follows the PRP approach.

- PRP, Chapter 2
- EP, Sections 4.1, 4.18, 5.1, and 7.1

The first 8 items are relevant for the quiz on Thursday, October 14. The remaining items are not.

1. Define "random variable," paying attention to the restriction in the definition that is imposed by the choice of σ-field. Give a couple of simple examples of random variables that can be associated with flipping 5 coins or rolling 2 dice.

2. Suppose that Ω is the sample space associated with flipping a coin twice. Make a table showing the values of each of the following random variables on this sample space. Notice that this table has nothing to do with whether the coin is fair or whether the tosses are independent.

   - X_1 is the number of heads.
   - X_2 is the amount you end up with if you start with $1, bet it on heads for the first flip, then bet everything you have on heads for the second flip.
   - X_3 is the amount you end up with if you borrow $1, bet it on heads for the first flip, then borrow $2 and bet it on heads for the second flip.
   - X_4 is the amount you end up with if you borrow $2, bet it on heads for the first flip, then bet all $4 on the second flip if you won, or borrow another $1 and bet it on heads for the second flip if you lost.

3. Sketch graphs of the discontinuous distribution functions associated with random variables X_2 and X_1 above.

4. As a simple example of a distribution function for a continuous random variable Z, suppose that you choose a random number uniformly in $[0, 1]$ and square it. Calculate $\mathbb{P}(Z \leq \frac{1}{4})$ and sketch the distribution function of Z.

5. As an example of a random variable that is strictly increasing on $(-\infty, \infty)$, consider the "Mesopotamia problem." A hiker is lost midway between two north-south rivers that are two miles apart. His compass is broken and he cannot see the sun, so he decides just to walk in a straight line in a random direction until he reaches a river. Random variable Y is the (signed) distance north or south of his starting point when he reaches a river.

6. Prove the three properties of the distribution function of a random variable listed in Lemma 6 on p. 28 of PRP. Make it clear, by referring to earlier examples, why the use of $<$ and $\leq$ is correct in (b) and why $F(x)$ need only be right-continuous.

7. Prove Lemma (11) on page 30 of PRP, which shows how to compute probabilities of most events of interest from a distribution function. Part (c) is tricky because the event of interest must be expressed as a countable intersection of intervals.

8. Explain what is meant by a discrete random variable and by a continuous random variable. Give one example of a random variable X that is neither discrete nor continuous because it is a mixture. As a realistic-sounding scenario, X could be the weight in ounces of a randomly-chosen package of cheese from a supermarket.

9. Define the indicator function of an event. Supposing that events A_1, A_2, and A_3 are "coin flips 1, 2, and 3 respectively are heads," make a table showing the functions I_1, I_2, I_3, and $S_3 = I_1 + I_2 + I_3$. Sketch a graph of the distribution function of $\frac{S_3}{3}$ for the case where the flips are independent and the probability of a head is $\frac{1}{2}$. Repeat for the case where the probability of a head is $\frac{1}{3}$. Show how this example illustrates identity (10) at the bottom of p. 29 of PRP.

10. As a preliminary to the proof of the law of averages, prove that the following are true for all real x:

    - $$x^2 + \frac{1}{4} \geq x$$
    - $$2e^{x^2} \geq e^x$$
    - $$f(x) = e^{x^2} + x - e^x \geq 0$$

    The strategy is to show that $f(x)$ has a minimum value of zero and that its second derivative is everywhere non-negative.

11. State the "law of averages" and prove it by deriving Bernstein's identity. This is nicely explained on pages 31 and 32 of PRP.

12. Using the first Borel-Cantelli lemma and Bernstein's identity, prove that the average of n independent Bernoulli variables each with probability p cannot exceed $p + \epsilon$ for infinitely many n. This is a "strong law of averages."

13. The Schwarzkopf attack.

    This is a highly contrived example of a "singular" random variable that is neither discrete nor continuous nor a mixture.

    Your task is to defend 1 mile of border against an attack masterminded by Gen. Norman Schwarzkopf, who achieved fame for his surprise move from Saudi Arabia into Iraq during the 1991 Gulf War. Fortunately, a spy has captured the orders to the attacking commander. Here they are.

    0. At noon, take up a position at $x = \frac{1}{2}$ miles.
    1. Toss a coin. If it comes up heads, move $\frac{1}{3}$ of a mile east to $x = \frac{5}{6}$ miles. If it comes up tails, move $\frac{1}{3}$ of a mile west to $x = \frac{1}{6}$ miles. This can be accomplished in half an hour.
    2. Toss a coin. If it comes up heads, move $\frac{1}{9}$ of a mile east. If it comes up tails, move $\frac{1}{9}$ of a mile west. This can be accomplished in a quarter of an hour.

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    n. Toss a coin. If it comes up heads, move $\frac{1}{3^n}$ of a mile east. If it comes up tails, move $\frac{1}{3^n}$ of a mile west. This can be accomplished in $\frac{1}{2^n}$ of an hour.

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    final. At 1 PM, attack!

    (a) Show that the infinite number of steps implied by the orders require 1 hour.

    (b) Show the the sum of the lengths of all the intervals where the attacking commander cannot be located is 1 mile, so that a defending unit that takes up a random position is almost certain to be in the wrong place.

    (c) Show that at 1 PM, the set of points where the attacking commander can be located is uncountably infinite. (Hint: Express the position x as a fraction in a base-3 number system. For example .202 means $\frac{2}{3} + \frac{2}{27}$, the position resulting from heads-tails-heads. There is now an obvious 1-to-1 correspondence with the real numbers between 0 and 1, which are uncountable.)

(d) Let $F(x)$ denote the probability that the attack occurs to the west of position x. Sketch a graph of $F(x)$, and argue that its derivative is zero almost everywhere but is undefined at uncountably many points.

14. Prove that the set of points x in a sample space Ω for which $\mathbb{P}(x) > 0$ is countable, even if Ω is uncountable.

    This proof is not in PRP, but it is easy. Just consider, for each positive n, the set of points for which

    $$\frac{1}{n+1} < \mathbb{P}(x) \leq \frac{1}{n}$$

    Each such set is finite, so you have a countable union of finite sets.

15. Give examples of random variables that can be associated with a dart board.